Many triangulated odd-dimensional spheres

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[1] joint with Eran Nevo, Stedman Wilson, arXiv:1408.3501

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- $\log s_3(n) \geq \Omega(n^{5/4})$ (Pfeifle-Ziegler, 2004).

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We improve the lower bound to:

Theorem 1

$$\log s_{2k-1}(n) \geq \Omega(n^k)$$

Many geodesic odd spheres

It is well known (Alon, Goodman-Pollack 1986) that
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$$\Omega(n^{k-1+\frac{1}{k}}) \leq \log g_{2k-1}(n) \leq O(n^k)$$

The case $d = 3$

For $d = 3$ ($k = 2$) we get

$$\Omega(n^{\frac{3}{2}}) \leq g_3(n) \leq O(n^2), \qquad \Omega(n^2) \leq s_3(n) \leq O(n^2 \log n).$$

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  This answers questions of Erickson and Ziegler.

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For arbitrary 3-spheres we manage to do this with $\alpha(n) \in \Theta(n)$.
For geodesic 3-spheres we do this with $\alpha(n) \in \Theta(\sqrt{n})$.

The join of two paths

Let $a_1, \ldots, a_n \in \ell_a := \{(t, 0, 1) : t \in \mathbb{R}\}$ and $b_1, \ldots, b_m \in \ell_b := \{(0, t, -1) : t \in \mathbb{R}\}$ and let $A = \{a_i\}_{i \in [n]} \cup \{b_j\}_{j \in [m]}$.

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The only triangulation of A is the *join of two paths*:

$$\mathcal{T} = \{a_i a_{i+1} b_j b_{j+1} : i = 1, \ldots, n-1,\ j = 1, \ldots, m-1\}.$$

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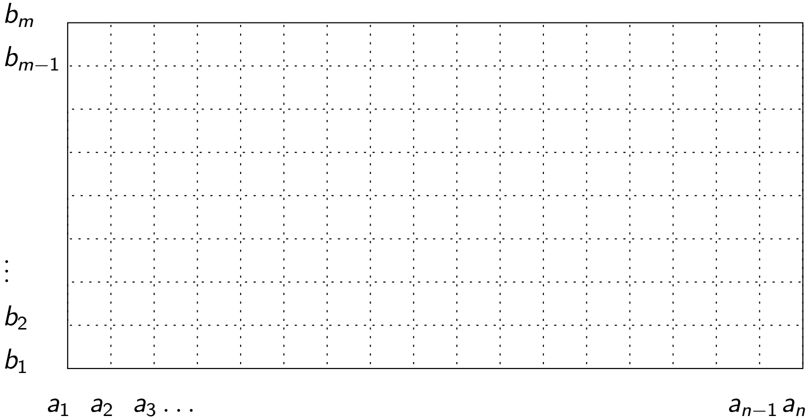
The combinatorics of $\mathcal{T}$ is very nicely encoded in an $[n-1] \times [m-1]$ grid, so that every subset of tetrahedra in $\mathcal{T}$ corresponds to a subset of $[n-1] \times [m-1]$.

Grid



Grid convexity

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1. $|\mathcal{T}_B|$ is a topological 3-ball (and shellable) if and only if B is grid-unimodal (B is strongly connected and $B \cap (\{i\} \times [m-1])$ and $B \cap ([n-1] \times \{j\})$ are connected (that is, intervals), $\forall i, j$).

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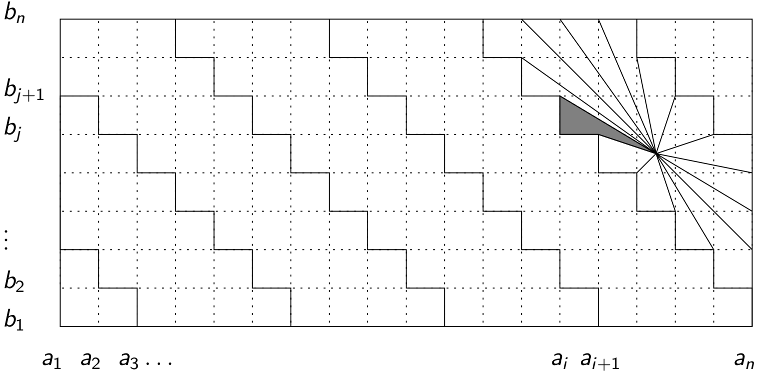
2. $|\mathcal{T}_B|$ is star convex from every point (equivalently, from some point) in the interior of the tetrahedron $T_{ij} \subset \mathcal{T}_B$ if and only if $\forall (i', j') \in B$ we have $[i, i'] \times [j, j'] \subset B$.

Construction 1: $2^{\Omega(n^2)}$ 3-spheres



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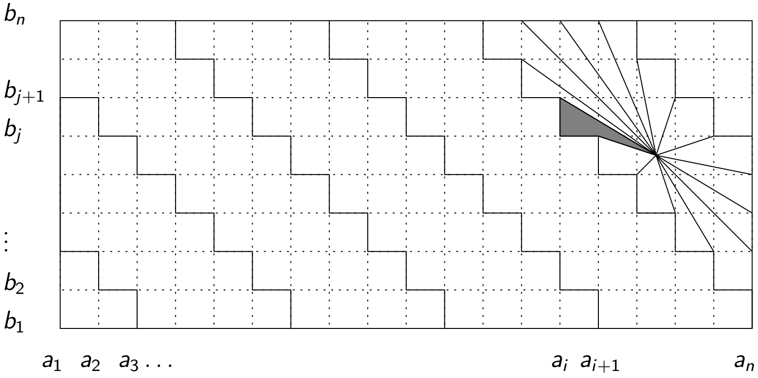

Theorem

There is a polyhedral 3-sphere with $5n/2 \pm O(1)$ vertices having $n^2/2 \pm O(N)$ bipyramids (refutes [Erickson 1999])

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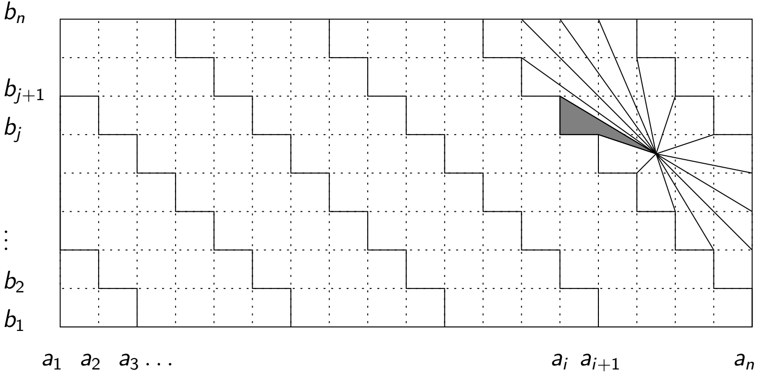

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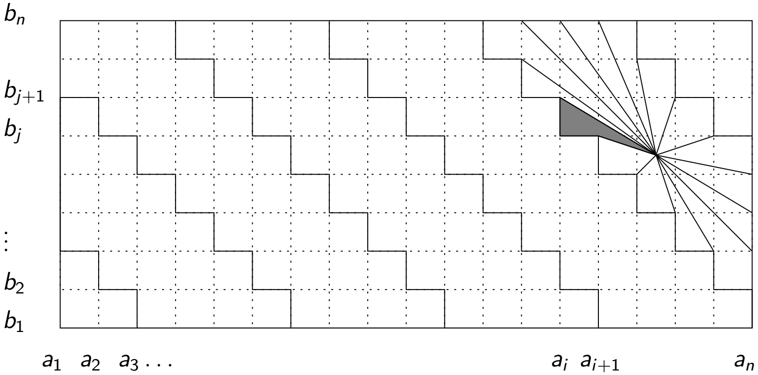
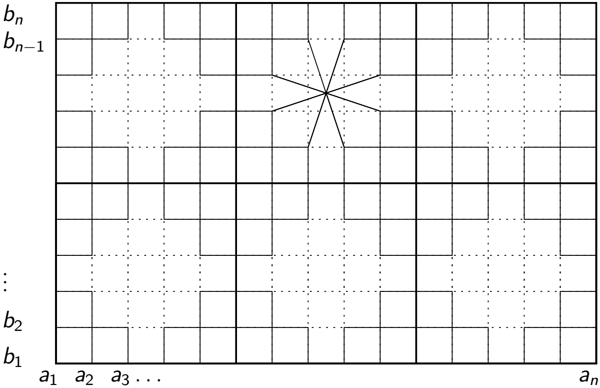


Corollary

There are at least $2^{\left(\frac{2N^2}{25} \pm O(N)\right)} \sim 1.0570^{N^2}$ triangulations of the 3-sphere with N vertices.

Construction 2: $2^{\Omega(n^{3/2})}$ geodesic 3-spheres



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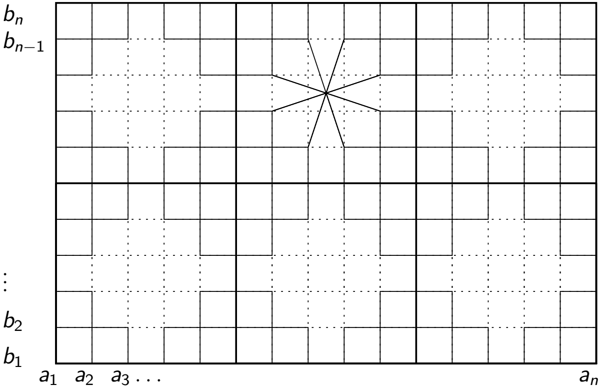

Let $n = m = kl + 1$ and divide the $[n-1]\times$ $[n-1]$ grid into l^2 subgrids of size $k \times k$.

The "aztec diamond" in each subgrid is a star-convex ball that can be subdivided into $2k - 2$ bipyramids.

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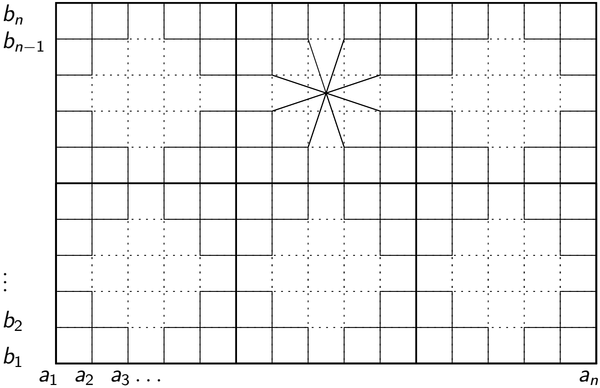

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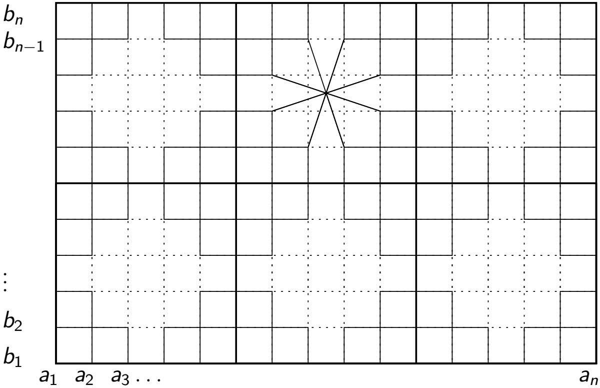

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Theorem

There is a geodesic 3-sphere with $3n \pm O(1)$ vertices consisting of $n^2/2 \pm O(n)$ tetrahedra and $2n^{3/2} \pm O(n)$ bipyramids.

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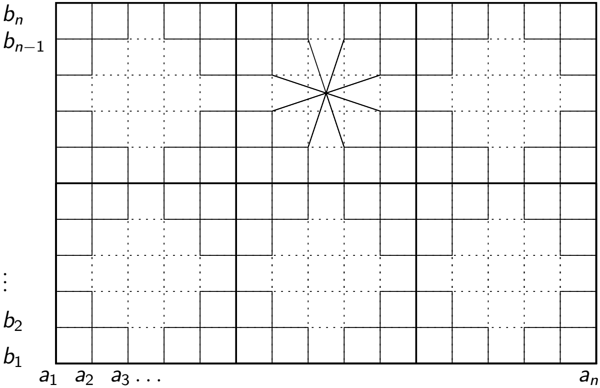

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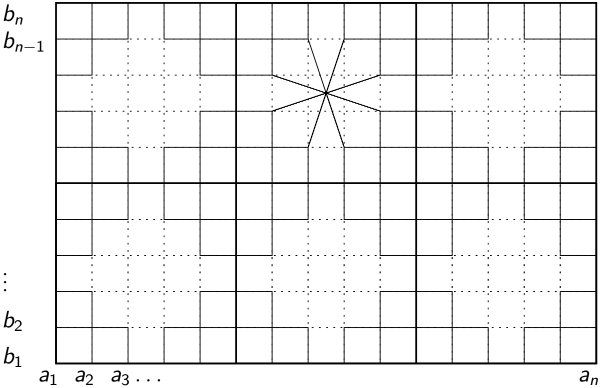
The "aztec diamond" in each subgrid is a star-convex ball that can be subdivided into $2k - 2$ bipyramids. Letting $k = l$:

Corollary

There are at least $2^{2(N/3)^{3/2} - O(N)}$ geodesic triangulations of the 3-sphere with N vertices.

Construction 2: $2^{\Omega(n^{3/2})}$ geodesic 3-spheres



Remark

The geodesic polyhedral 3-sphere obtained from the aztec diamonds is *regular* (a.k.a. weighted Delaunay), which means it is part of a polytopal 3-sphere.

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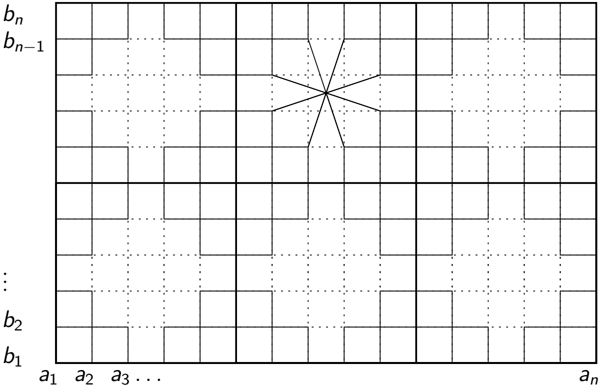

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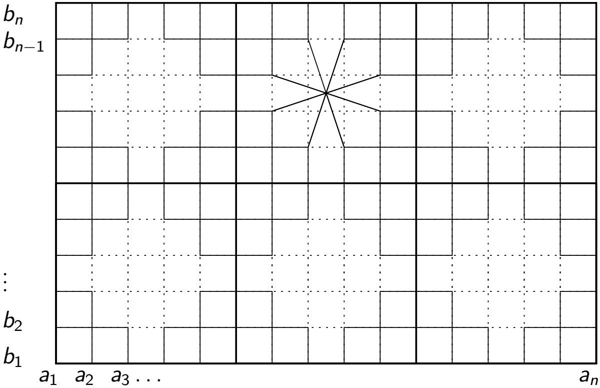

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The triangulation of it obtained refining every bipyramid into 3 tetrahedra is also regular.

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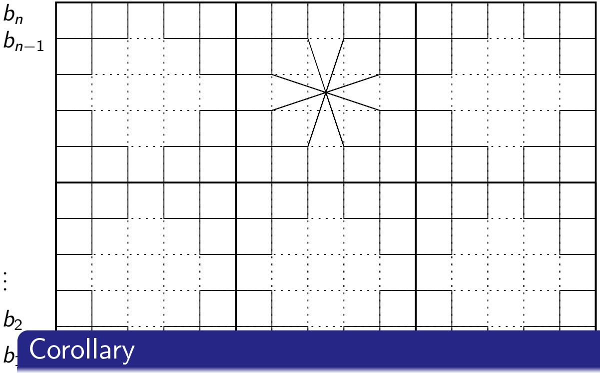

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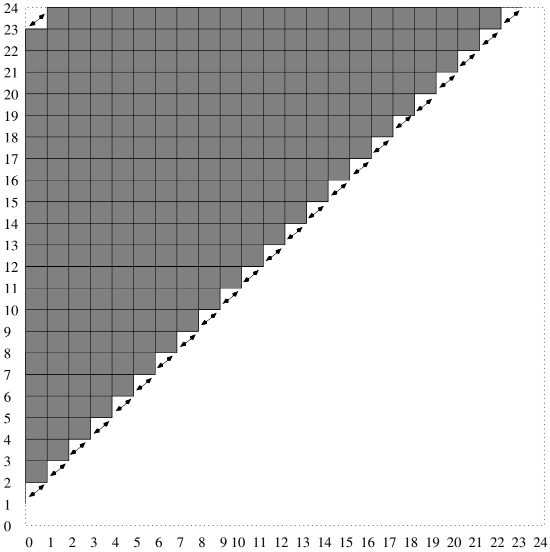
Corollary

There are simplicial 4-polytopes with N vertices having $2(N/3)^{3/2} - O(N)$ edges of degree 3 (or, there are simple 4-polytopes with N vertices having $2(N/3)^{3/2} - O(N)$ triangular 2-faces). (Answers a question by Ziegler)

Cyclic polytopes do slightly better



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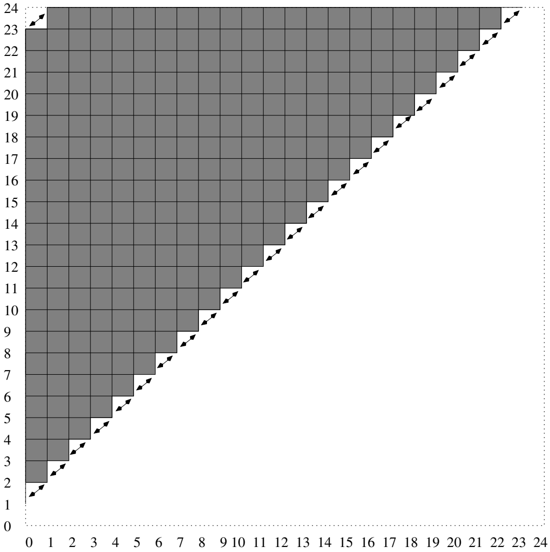

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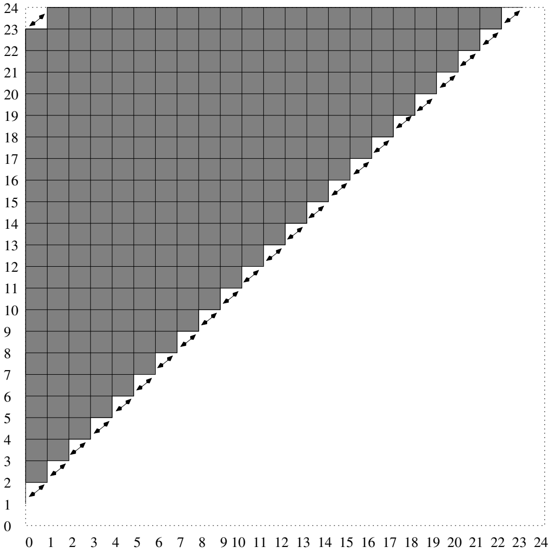

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There are at least $2^{\left(\frac{4N^2}{25} \pm O(N)\right)} \sim 1.117^{N^2}$ triangulations of the 3-sphere with N vertices.

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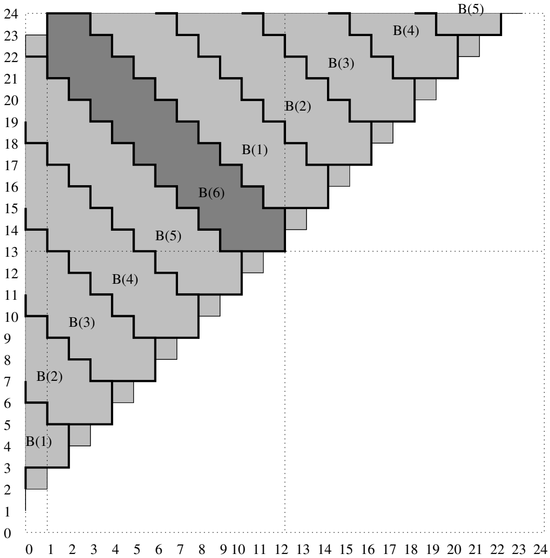

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Higher d

More or less the same ideas work in higher odd dimension $2k-1$, taking the join of k paths instead of 2.

Theorem

There are at least $2^{\left(\frac{2}{3k^{k+1}}\right)N^k}$ PL $(2k-1)$-spheres on N vertices.

Theorem

There are $2^{\frac{2}{(k-1)!(k+1)^k}N^{k-1+\frac{1}{k}}}$ geodesic triangulations of the $(2k-1)$-sphere.

The end

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THANK YOU