# Bounds on equivariant Betti numbers for symmetric semi-algebraic sets

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## Introduction

Let $\mathbf{R}$ be a real closed field (whose algebraic closure is $\mathbf{C}$ ). For any finite set $\mathcal{P} \subset \mathbf{R}[X_1, \ldots, X_k]$ (respectively, $\mathcal{P} \subset \mathbf{C}[X_1, \ldots, X_k]$), we denote by $\mathrm{Zer}(\mathcal{P}, \mathbf{R}^k)$ (respectively $\mathrm{Zer}(\mathcal{P}, \mathbf{C}^k)$) the set of common zeros of $\mathcal{P}$ in $\mathbf{R}^k$(respectively $\mathbf{C}^k$). For a finite set $\mathcal{P} \subset \mathbf{R}[X_1, \ldots, X_k]$ a $\mathcal{P}$ -semi-algebraic set is a semi-algebraic subset of $\mathbf{R}^k$ defined by a quantifier-free formula (resp. formula without negations) with atoms of the form $P\{<,>,=\}0$ (resp. with $P \in \mathcal{P}$).
Let $\mathbf{k} = (k_1, \ldots, k_\omega) \in \mathbb{Z}^\omega_{>0}, k = \Sigma_{i=1}^\omega k_i$ and denote

$$S_{\mathbf{k}} = S_{k_1} \times \cdots \times S_{k_\omega}.$$

We consider $X$ to be a semi-algebraic subset of $\mathbf{R}^{\mathbf{k}}$ or a constructible subset of $\mathbf{C}^{\mathbf{k}}$, such that the product of symmetric groups $S_{\mathbf{k}}$ acts on $X$ by separately permuting each block of coordinates. We denote by $X/S_{\mathbf{k}}$ the *orbit space* of this action. If $\mathrm{char}(\mathbb{F}) = 0$, then $H^*(X/S_{\mathbf{k}},\ F)$ is canonically isomorphic to the equivariant cohomology groups $H^*_{S_{\mathbf{k}}}(X, \mathbb{F})$ (defined using the Borel construction). Therefore we can refer to $b_i(X/S_{\mathbf{k}}, \mathbb{Q})$ as the equivariant Betti numbers with coefficients in $\mathbb{Q}$.

## Classical bounds on Betti numbers

The question of bounding the sum of the Betti numbers of real varieties has a long history starting with the work of Oleĭnik and Petrovskiĭ, Thom and Milnor who gave the following bound:

$$\Sigma_i\, b_i(\mathrm{Zer}(\mathcal{P}, \mathbf{R}^k), \mathbb{F}) \leq d(2d-1)^{k-1} = (O(d))^k.$$

By taking real and imaginary parts one can derive also a bound in the complex case:

$$\Sigma_i\, b_i(\mathrm{Zer}(\mathcal{P}, \mathbf{C}^k), \mathbb{F}) \leq d(2d-1)^{2k-1} = (O(d))^{2k}$$

In the semi-algebraic case, Basu, Pollack, Roy showed that for a $\mathcal{P}$-semi-algebraic set $S \subset \mathbf{R}^k$

$$\Sigma_i\, b_i(S, \mathbb{F}) = \Sigma_{i=0}^k \Sigma_{j=1}^{k-i} \binom{s}{j} 6^j d(2d-1)^{k-1} = (O(sd))^k,$$

where $s := |\mathcal{P}|$, $d := \max_{p \in \mathcal{P}} \deg p$.
Notice that these bounds are exponential in $k$, for fixed $d$ and also that these bounds are tight.
However, in special situations, there are better bounds. For example Barvinok showed that if a semi algebraic set $S \subset \mathbf{R}^k$ is defined by polynomials $Q_1 \geq 0, \ldots, Q_m \geq 0$ which are at most of degree 2, then

$$\Sigma_i\, b_i(S, \mathbb{F}) \leq k^{O(m)}.$$

## Symmetric complex varieties

Let $\mathcal{P} \in \mathbf{C}[X_1, \ldots, X_k]$ be symmetric in $X_1, \ldots, X_k$ such that $\deg(p) \leq d$ for all $p \in \mathcal{P}$. Then the quotient space $V_{\mathbf{C}}/S_k$ is an algebraic subset of $\mathbf{C}^d$ in the following way: For each $p \in \mathcal{P}$, there exists $G_p \in \mathbf{C}[Z_1, \ldots, Z_d]$ with $\deg G_p \leq d$, such that $p = G_p(e^{(k)}, \ldots, e_d^{(k)})$, where $e_i^{(k)} \in \mathbb{Z}[X_1, \ldots X_k]$ denotes the i-th elementary symmetric polynomial. The quotient space, $V_{\mathbf{C}}/\mathcal{S}_k$ , is then homeomorphic to $\mathrm{Zer}(\mathcal{G}; \mathbf{C}^d)$. Therefore we find:
For any field of coefficients $\mathbb{F}$,

$$\begin{aligned} b\,(V_{\mathbf{C}}/S_k, \mathbb{F}) &\leq 2d(4d-1)^{2d-1} \\ &= d^{O(d)}. \end{aligned}$$

This bound is independent of $k$ for $k > d$.

## A motivational real example

Consider

$$P = \Sigma_{i=1}^k \left( \prod_{j=1}^d (X_i - j) \right)^2.$$

Let $V_{\mathbf{R}} = \mathrm{Zer}\left(\{P\}, \mathbf{R}^k\right)$ then $V_{\mathbf{R}}$ and $V_{\mathbf{R}}/S_k$ are zero-dimensional and we have $b_0(V_{\mathbf{R}}, \mathbb{Q}) = d^k$. Further $b_0\,(V_{\mathbf{R}}/S_k, \mathbb{Q}) = \Sigma_{\ell=1}^d p(k, \ell)$, where $p(k, \ell)$ denotes the number of partitions of $k$ into $\ell$ parts. It follows that his zero-dimensional variety $V = \{1, \ldots, d\}^k$ cannot be defined by any finite set of symmetric complex polynomials of degree bounded by $d$ (for $k > d$) unlike in the real case.

## Main result

Let $\mathbf{k} = (k_1, \ldots, k_\omega) \in \mathbb{Z}^\omega_{>0}, k = \Sigma_{i=1}^\omega k_i$. Let $P \in \mathbf{R}[\boldsymbol{X}^{(1)}, \ldots, \boldsymbol{X}^{(\omega)}]$, where each $\boldsymbol{X}^{(i)}$ is a block of $k_i$ variables, be a non-negative polynomial, such that $V = \mathrm{Zer}\left(P, \mathbf{R}^k\right)$ is invariant under the action of $S_{\mathbf{k}}$ permuting each block $\boldsymbol{X}^{(i)}$ of $k_i$ coordinates. Let $\deg_{X^{(i)}}(P) \leq d$ for $1 \leq i \leq \omega$. Then, for any field of coefficients $\mathbb{F}$,

$$b(V/S_{\mathbf{k}}, \mathbb{F}) \leq \sum_{1 \leq \ell_i \leq \min(2d, k_i)} p(\mathbf{k}, \ell_1, \ldots, \ell_\omega) d(2d-1)^{|\mathbf{l}|+1}.$$

Moreover, for all $i \geq \Sigma_{i=1}^\omega \min(2d, k_i)$

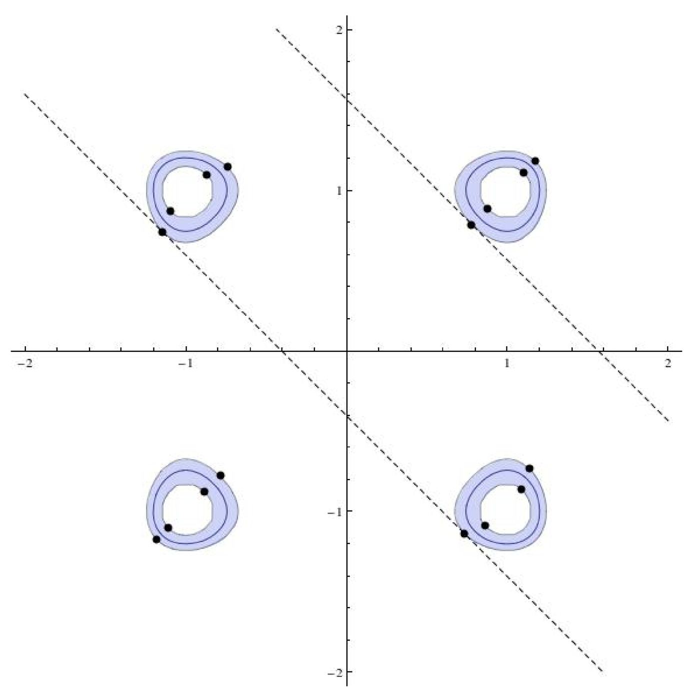
$$b_i(V/S_{\mathbf{k}}, \mathbb{F}) = 0.$$

## Tools for the proof (1)

We argue using Morse-theory. For this we need to define an **equivariant deformation** and use **equivariant Morse theory**.
Consider the example of the polynomial

$$P = (X_1^2 - 1)^2 + (X_2^2 - 1)^2.$$

The set $\mathrm{Zer}\,(P, \mathbf{R}^2)$ is topologically 4 circles.



Then the basic semi-algebraic set $S = \{x \in \mathbf{R}\langle\zeta\rangle^2 \mid \bar{P} \leq 0\}$, where

$$\bar{P} = \mathrm{Def}(P, \zeta) = (X_1^2 - 1)^2 + (X_2^2 - 1)^2 - \zeta,$$

is topologically 4 annuli and homotopy equivalent to $\mathrm{Zer}\,(P, \mathbf{R}^2)$

## Tools for the proof (2)

We show that $b(V/S_{\mathbf{k}}, \mathbb{F})$ **is bounded by the number of critical values of** $e_1^{(\mathbf{k})}$ **restricted to** $\mathrm{Zer}(\bar{P}, \mathbf{R} < \zeta >^{\mathbf{k}})$. Moreover, there is a recipe for measuring the exact change in the homotopy type of $V/S_{\mathbf{k}}$ as we cross each critical level from the local data at any one of the critical point at that level.
Further, we **restrict the possible orbits of critical points**: For any pair $(\mathbf{k}, \mathbf{l})$, where $\mathbf{k} = (k_1, \ldots, k_\omega) \in \mathbb{Z}^\omega_{>0}$, $k = \Sigma_{i=1}^\omega k_i$, and $\mathbf{l} = (\ell_1, \ldots, \ell_\omega)$, with $1 \leq \ell_i \leq k_i$, we denote by $A_{\mathbf{k}}^{\mathbf{l}}$ the subset of $\mathbf{R}^{\mathbf{k}}$ defined by

$$A_{\mathbf{k}}^{\mathbf{l}} = \left\{x = (x^{(1)}, \ldots x^{(\omega)}) \mid \mathrm{card}\left(\bigcup_{j=1}^{k_i} \{x_j^{(i)}\}\right) = \ell_i\right\}.$$

Then each critical point of $e_1^{(\mathbf{k})}$ restricted to $V$ is contained in $A_{\mathbf{k}}^{\mathbf{l}}$ for some $\mathbf{l} = (\ell_1, \ldots, \ell_\omega)$ with each $\ell_i \leq d$.
The efficient algorithms for computing the generalized Euler-Poincaré characteristic of symmetric varieties, as well as the vanishing results, follow from this lemma and the bound on the number of critical values on the next two slides.

## Betti Numbers of Projections

Let $\mathcal{P} \in \mathbf{R}[Y_1, \ldots, Y_m, X_1, \ldots, X_k]$ be a family of polynomials with $\deg(P) \leq d$ and let $S$ be a $\mathcal{P}$ semi algebraic set, which is closed and bound.
Denote by $\pi : \mathbf{R}^{m+k} \longrightarrow \mathbf{R}^m$ be the projection map to the first $m$ co-ordinates. Gabrielov, Vorobjov, Zell showed that

$$b(\pi(S), \mathbb{Q}) \leq \sum_{0 \leq p < m} b(\underbrace{S \times_\pi \cdots \times_\pi S}_{p+1}, \mathbb{Q}),$$

where $\underbrace{S \times_\pi \cdots \times_\pi S}_{p+1}$ denotes th $p$-fold fibered product of $S$. In conjunction with the general bounds on the sum of Betti numbers their results yields

$$b(\pi(S), \mathbb{Q}) \leq O(d)^{(k+1)m}.$$

The group $S_{p+1}$ acts on the p-fold fibered product. Let $Sym_\pi^{(p)}(X)$ denote the quotient $\underbrace{X \times_\pi \cdots \times_\pi X}_{p+1}/S_{p+1}$.
We show that

$$b(\pi(S), \mathbb{F}) \leq \Sigma_{0 \leq p < m}\, b(Sym_\pi^{(p)}(S), \mathbb{F}).$$

This then implies

$$b(\pi(S), \mathbb{Q}) \leq O(d)^{m + O(d^k)}.$$

## Outlook

In a forthcoming paper we study the decomposition of the $i$-th cohomology group $\mathrm{H}^i(S, \mathbb{F})$ in terms of irreducible representations. Also in this case we can show that the mulitplicities of the irreducible representations can be bounded by a polynomial in $k$.

## References

1 S. Basu, C. Riener, *Bounding the equivariant Betti numbers and computing the generalized Euler-Poincaré characteristic of symmetric semi-algebraic sets*, `arXiv:1312.6582`.
2 S. Basu, C. Riener, *On the isotypic decomposition of homology modules of symmetric semi-algebraic sets*, in preparation.

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