Combinatorial and topological models for spaces of schedules

Martin Raussen

Department of Mathematical Sciences, Aalborg University, Denmark

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Discrete, Computational and Algebraic Topology

Department of Mathematical Sciences

Copenhagen University

TOC

Acknowledgements

Contributions by Lisbeth Fajstrup (AAU, DK), Éric Goubault (École Polytechnique Paris, F), Roy Meshulam (Technion, Haifa, IL), Krysztof Ziemiański (Warsaw, PL), ...

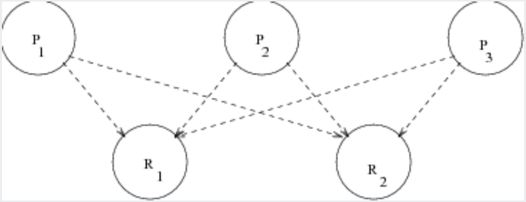
Concurrency

- In computer science, concurrency is a property of systems in which several computations are executing simultaneously and potentially interacting with each other.
- The computations may be executing on multiple cores in the same chip, in time-shared threads on the same processor, or executed on physically separated processors.
- A number of mathematical models have been developed for general concurrent computation including Petri nets and process calculi.
- Main interest here: Specific applications tuned to static program analysis – design of automated tools to test correctness etc. of a concurrent program regardless of specific timed execution.

A simple-minded approach to concurrency
Avoid access collisions

Access collisions

may occur when n processes P_i compete for m resources R_j.



Only κ (capacity) processes can be served at any given time.

Tool: Semaphores

Semantics: A processor has to lock a resource and to relinquish the lock later on!

Description/abstraction: $P_i : \ldots PR_j \ldots VR_j \ldots$ (E.W. Dijkstra)

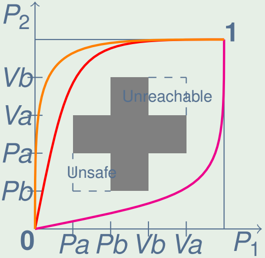
P: probeer; V: verhoog

Schedules in "progress graphs"

One semaphore on a time line

$$0 \longrightarrow P_a \longrightarrow P_b \longrightarrow V_a \longrightarrow V_b \longrightarrow 1$$

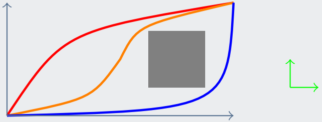
Two semaphores: The Swiss flag example



PV-diagram from
P_1 : $P_aP_bV_bV_a$
P_2 : $P_bP_aV_aV_b$

Executions are directed paths – since time flow is irreversible – avoiding a forbidden region (shaded).
Dipaths that are dihomotopic (through a 1-parameter deformation consisting of dipaths) correspond to equivalent executions.
Deadlocks, unsafe and unreachable regions may occur.

Directed paths in state spaces

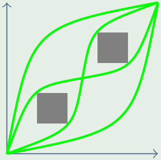


- A state space with “hole(s)”
- Paths from a start point to an end point with preferred direction: dipaths
- 1-parameter deformations of dipaths: dihomotopies

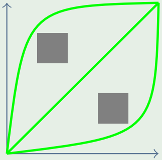
First observation

Homeomorphic state spaces may admit different types of dipaths (up to deformation):

1. 4 classes:



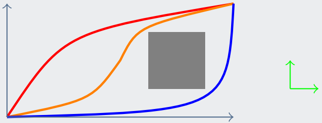
2. 3 classes:



Objects of study: Spaces with directed paths

First Example for impact of directedness

Directed paths in state spaces



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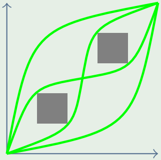
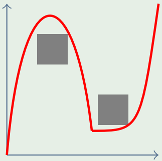


2. "forbidden" class:



Directed topology: The twist has a price

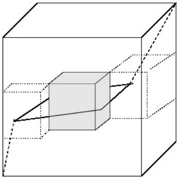
2nd observation: Neither homogeneity nor cancellation nor group structure

Question

Can methods from algebraic topology shed light on the space $\vec{P}(X)(\mathbf{x}_0, \mathbf{x}_1)$ of directed paths – execution space – in the state space X from $\mathbf{x}_0$ to $\mathbf{x}_1$?

Problem: Symmetry breaking

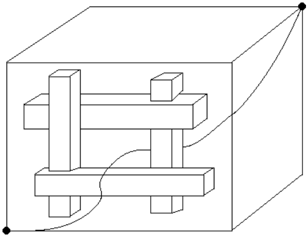
The reverse of a dipath need not be a dipath.
$\rightsquigarrow$ less structure on algebraic invariants.



"Birth and death" of dihomotopy classes

Directed topology

Loops do not tell much; concatenation ok, cancellation not!
Replace group structure by category structures!
Example: Fundamental category $\vec{\pi}_1(X)$ – admitting a van Kampen theorem.



A dipath that is homotopic but not dihomotopic to a dipath on the boundary of the cube

- Such a deformation exists but:
- Every deformation will violate directedness.
- How to prove this?
- Remark: Need at least 3D-models for such an example!
- Space of dipaths in example $\simeq (S^1 \vee S^1) \sqcup *$.

State space $X \rightsquigarrow$ path category $\vec{P}(X)$

State spaces – three main cases of interest

- $X \subset \mathbf{R}^n$ a Euclidean cubical complex – cut out a forbidden region F consisting of hyperrectangular holes
- $X \subset \prod_i \Gamma_i$, a product of directed graphs with cubical holes (allowing branches and directed loops)
- X a directed cubical complex[a] (with directed loops): a Higher Dimensional Automaton (with labels)

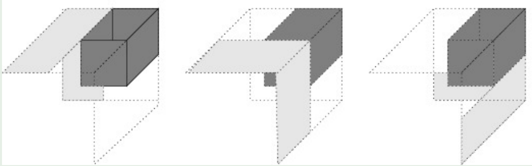
[a] as in geometric group theory

From state space X to path space $\vec{P}(X)(\mathbf{x}_0, \mathbf{x}_1)$

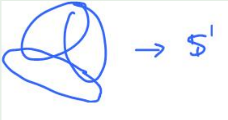
Challenge: Provide path spaces with a combinatorial (simplicial) structure

Simplicial models for spaces of dipaths

A cover of the path space associated to the “floating cube”



Cover: Dipaths through the lightgrey areas



Cover giving rise to $\partial\Delta^2 \cong S^1$

Theorem (R; 2010)

Let X be a state space consisting of a cube $\square^n$ from which l hyperrectangles are removed.
The space $\vec{P}(X)(\mathbf{0},\mathbf{1})$ of dipaths in X from bottom $\mathbf{0}$ to top $\mathbf{1}$ is homotopy equivalent to the nerve of a category $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$.
This category has a geometric realization as a prodsimplicial complex $\mathbf{T}(X)(\mathbf{0},\mathbf{1}) \subset (\partial\Delta^{n-1})^l$ – its building blocks are products of simplices.

Tool: Subspaces of state space X and of $\vec{P}(X)(\mathbf{0},\mathbf{1})$

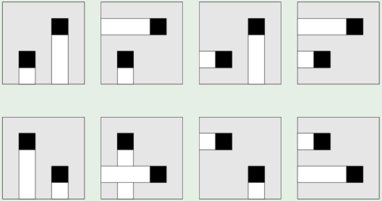
$X = \vec{I}^n \setminus F$, $F = \bigcup_{i=1}^{l} R^i$; $R^i =]\mathbf{a}^i, \mathbf{b}^i[$; $\mathbf{0}, \mathbf{1}$ the two corners in I^n.

Definition (Restricted state spaces)

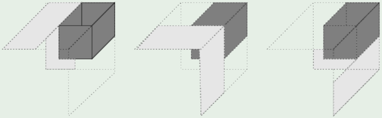
1. $X_{ij} = \{x \in X \mid x \leq \mathbf{b}^i \Rightarrow x_j \leq a^i_j\}$ – *direction j restricted at hole i*
2. M a binary $l \times n$-matrix: $X_M = \bigcap_{m_{ij}=1} X_{ij}$ – *Which directions are restricted at which hole?*

Examples: two holes in 2D – one hole in 3D (dark)

$M =$ $\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$ $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ $\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}$



$M =$ $[100]$ $[010]$ $[001]$



$M = [111]$: No d-path!

Covers by contractible (or empty) subspaces

Bookkeeping with binary matrices

Binary matrix posets

$M_{l,n}$ poset ($\leq$) of binary $l \times n$-matrices

$M_{l,n}^*$ no row vector is the zero vector – every hole obstructed in at least one direction

Theorem (A cover by contractible subspaces)

1. $$\vec{P}(X)(\mathbf{0},\mathbf{1}) = \bigcup_{M \in M_{l,n}^*} \vec{P}(X_M)(\mathbf{0},\mathbf{1}).$$

2. *Every path space $\vec{P}(X_M)(\mathbf{0},\mathbf{1})$, $M \in M_{l,n}^*$, is empty or contractible.* *Which is which? Deadlocks!*

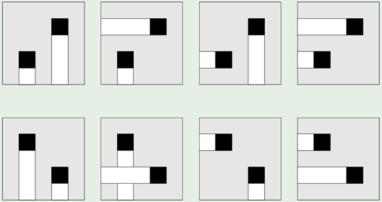
Proof.

(2) Subspaces X_M, $M \in M_{l,n}^*$ are closed under $\vee$ = l.u.b. $\square$

Combinatorics:
Poset category
$\mathcal{C}(X) \subseteq M^*_{l,n}$ consists of "**alive**" matrices M with $\vec{P}(X_M) \neq \emptyset$ – no deadlock!

Topology:
Prodsimplicial complex
$\mathbf{T}(X) \subseteq (\Delta^{n-1})^l$ colimit of $\Delta_M = \Delta_{m_1} \times \cdots \times \Delta_{m_l} \subseteq \mathbf{T}(X)$ M alive – one simplex Δ_{m_i} for every hole.

Examples of path spaces



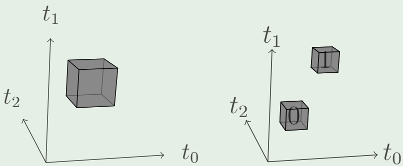
$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \supset \mathcal{C}(X)$$

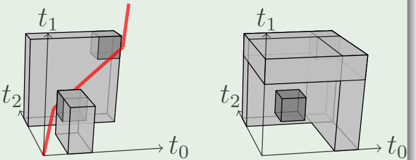
- $\mathbf{T}(X_1) = (\partial\Delta^1)^2 = 4*$
- $\mathbf{T}(X_2) = 3*$ – deadlock!

Further examples

State spaces, "alive" matrices and path spaces

1. $X = \vec{I}^n \setminus \vec{J}^n$
   - $\mathcal{C}(X) = M^*_{1,n} \setminus \{[1,\ldots,1]\}$.
   - $\mathbf{T}(X) = \partial\Delta^{n-1} \simeq S^{n-2}$.
2. $X = \vec{I}^n \setminus (\vec{J}^n_0 \cup \vec{J}^n_1)$
   - $\mathcal{C}(X) = M^*_{2,n} \setminus$ matrices with a $[1,\ldots,1]$-row.
   - $\mathbf{T}(X) \simeq S^{n-2} \times S^{n-2}$.





$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

alive dead(lock)

Theorem (A variant of the nerve lemma)

$\vec{P}(X) \simeq \Delta\mathcal{C}(X) \simeq \mathbf{T}(X).$

allows (in principal) to calculate homology,...

Proof.

- Functors $\mathcal{D}, \mathcal{E}, \mathcal{T} : \mathcal{C}(X)^{(\mathrm{op})} \to \mathbf{Top}$:
  $\mathcal{D}(M) = \vec{P}(X_M)$,
  $\mathcal{E}(M) = \Delta_M$,
  $\mathcal{T}(M) = *$
- $\mathrm{colim}\ \mathcal{D} = \vec{P}(X)$, $\mathrm{colim}\ \mathcal{E} = \mathbf{T}(X)$, $\mathrm{hocolim}\ \mathcal{T} = \Delta\mathcal{C}(X)$.
- The trivial natural transformations $\mathcal{D} \Rightarrow \mathcal{T}, \mathcal{E} \Rightarrow \mathcal{T}$ yield:
  $\mathrm{hocolim}\ \mathcal{D} \simeq \mathrm{hocolim}\ \mathcal{T}^* \simeq \mathrm{hocolim}\ \mathcal{T} \simeq \mathrm{hocolim}\ \mathcal{E}$.
- Segal's projection lemma:
  $\mathrm{hocolim}\ \mathcal{D} \simeq \mathrm{colim}\ \mathcal{D}$, $\mathrm{hocolim}\ \mathcal{E} \simeq \mathrm{colim}\ \mathcal{E}$.

□

2. approach: Towards configuration spaces

One semaphore

Path space captured by "times" $0 < t^1 < t^2 < t^3 < t^4 < 1 \in \mathring{\Delta}_4$

n semaphores

- A directed path (n threads) is encoded by $(t_1^1, \dots, t_1^{2k_1}; \dots; t_n^1, \dots, t_n^{2k_n}) \in \prod_1^n \mathring{\Delta}_{2k_i}$
- Forbidden dipaths: Successive P_a, V_a corresponding to $t_k^{i_a}, t_k^{j_a}$.

  Capacity $n-1$: $\max_{k=1}^n t_k^{i_a} < \min_1^n t_k^{j_a}$

  Capacity κ: $\max_{1 \le k_1 < \dots < k_{\kappa+1} \le n} t_{k_j}^{i_a} < \min_{1 \le k_1 < \dots < k_{\kappa+1} \le n} t_{k_j}^{j_a}$
- The space of all forbidden dipaths A corresponds to union of a bunch of subspaces $A_{\mathbf{i},\mathbf{j}}^{\kappa+1}(a)$ of type "$\max < \min$" within $\prod_1^n \mathring{\Delta}_{2k_i}$
- Path space as configuration space: $D = \prod_1^n \mathring{\Delta}_{2k_i} \setminus A$.

Subspace arrangments

A finite set of A of subspaces in affine or projective space.
Aim: To infer (topological) properties of the complement $M(A)$ from the intersection semilattice $L(A)$, partially ordered by containment.

Configuration spaces

1. $M_n(X) = \{x_1, \dots, x_n) \in X^n | \ i \neq j \Rightarrow x_i \neq x_j\}$; the complement of $A_n(X) = \bigcup_{i \neq j}\{x_i = x_j\}$ in X^n.
2. No-k-equal space $M_n^{(k)}(X)$ the complement of $A_n^{(k)}(X) = \bigcup_{1 \leq i_1 < \dots < i_k \leq n}\{x_{i_1} = \dots = x_{i_k}\}$.
3. $M_n^{(n)}(\mathbf{R}) = \mathbf{R}^n \setminus \Delta(\mathbf{R}) \simeq S^{n-2}$.
4. $M_n^{(k)}(\mathbf{R}) \subset \mathbf{R}^n$: no-$k$-equal space. Homology determined by Björner & Welker (1995); concentrated in dimensions $s(n-2)$. Cell structure and cohomology ring determined by Baryshnikov.

Path conf. spaces vs. subspace arrangements
(Dis-)similarities

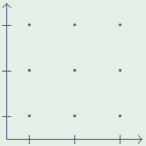
Comparison

- Path configuration space $D \subset \mathring{\Delta}_{\mathbf{k}}$ – not Euclidean (or projective).
- Complement of solutions of inequalities
- Still: Intersection semilattice matters!

A particular case: $Pa = Va$

Instantaneous use of resources. In this case:
Forbidden dipaths correspond to regions given by
equations $x_{j_1}^{i_1} = \cdots = x_{j_k}^{i_k}$.

Example: Time of access for $9 \cdot$ obstructions, $n = k = 2$

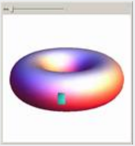


$\rightsquigarrow$ "Time space": $\mathring{\Delta}_3 \times \mathring{\Delta}_3 \setminus A \subset \mathbf{R}^6$ with
$A = \{(\mathbf{s}, \mathbf{t}) \in \mathring{\Delta}_3 \times \mathring{\Delta}_3 | \; s_i = t_j, 1 \leq i, j \leq 3\} \subset \mathbf{R}^6$ – difficult to draw!

And in higher dimensions?

Example: Dipaths on torus – with directed loops!

Torus with hole (1. approach: R - Ziemiański)



Dipaths in covering of torus with hole
$\rightsquigarrow$ state space $X_n = \mathbf{R}^n \setminus (\frac{1}{2} + \mathbf{Z}^n)$ and of dipaths with multidegree $\mathbf{k}$ in
$Z(\mathbf{k}) := \vec{P}(X_n)(\mathbf{0}, \mathbf{k}),\ \mathbf{k} \in \mathbf{Z}^n$

Definition (Multiindices generate polyn. ring and quadr. ideal)

- $\mathbf{l} = (l_1, \dots, l_n) \ll (m_1, \dots, m_n) = \mathbf{m} \in \mathbf{Z}_+^n \Leftrightarrow l_j < m_j, 1 \le j \le n.$
- $\mathcal{O}^n = \{(\mathbf{l}, \mathbf{m}) \mid \mathbf{l} \ll \mathbf{m} \text{ or } \mathbf{m} \ll \mathbf{l}\} \subset \mathbf{Z}_+^n \times \mathbf{Z}_+^n$ – ord. pairs
- $\mathbf{B}(\mathbf{k}) := \mathbf{Z}_+^n(\le \mathbf{k}) \times \mathbf{Z}_+^n(\le \mathbf{k}) \setminus \mathcal{O}^n$ – unordered pairs
- $\mathcal{I}(\mathbf{k}) := < \mathbf{lm} \mid (\mathbf{l}, \mathbf{m}) \in \mathbf{B}(\mathbf{k}) > \le \mathbf{Z}[\mathbf{Z}_+^n(\le \mathbf{k})]$
  quadratic ideal in graded polynomial ring – to cancel out!

Theorem (Cohomology and homology; R.-Ziemiański, 2014)

For $n > 2$, $H^*(Z(\mathbf{k})) = \mathbf{Z}[\mathbf{Z}_+^n(\le \mathbf{k})]/_{\mathcal{I}(\mathbf{k})}$.
All generators have degree $n-2$.
$H_*(Z(\mathbf{k})) \cong H^*(Z(\mathbf{k}))$ as abelian groups

Path spaces $Z(\mathbf{k}) = \vec{P}(X_n)(\mathbf{0}, \mathbf{k})$ as homotopy colimits

Index category $\mathcal{J}(n)$

Poset category of proper non-empty subsets of $[1:n]$ with inclusions as morphisms.
Via characteristic functions isomorphic to the category of non-identical binary vectors of length n: $\varepsilon = [\varepsilon_1, \dots \varepsilon_n] \in \mathcal{J}(n)$.
Classifying space (= nerve): $B\mathcal{J}(n) \cong \partial\Delta^{n-1} \cong S^{n-2}$.

Theorem

$Z(\mathbf{k}) \simeq \mathrm{hocolim}_{\varepsilon \in \mathcal{J}(n)} Z(\mathbf{k} - \varepsilon)$.

Proof of homology result

Inductive (Bousfield-Kan) spectral sequence argument, using projectivity of the functor $H_* : \mathcal{J}(n) \to \mathbf{Ab}_*, \ \mathbf{k} \mapsto H_*(Z(\mathbf{k}))$.

Theorem (K. Ziemiański: Surprising consequence of the technique)

Every finite simplicial complex occurs as homotopy type of a semaphore path space!!

Experiment: Configuration spaces and wedge lemma

- Configuration space for $Z(\mathbf{k})$:
  $D(\mathbf{k}) := \mathring{\Delta}_{\mathbf{k}} \setminus A(\mathbf{k}) = \mathring{\Delta}_{k_1} \times \cdots \times \mathring{\Delta}_{k_n} \setminus A(\mathbf{k}) \subset \widehat{\mathring{\Delta}}_{\mathbf{k}} \cong S^{|\mathbf{k}|}$ with $A(\mathbf{k}) = \bigcup_{1 \le i_j \le k_j} \{x^1_{i_1} = \cdots = x^n_{i_n}\}$ within compactification.
- (Co-)homology of $\widehat{A}(\mathbf{k}) \subset \widehat{\mathring{\Delta}}_{\mathbf{k}} = S^{|\mathbf{k}|}$ using the intersection poset Q of the cover defined by $A(\mathbf{k}) \rightsquigarrow$ Alexander duality $H_*(D(\mathbf{k}))$

Application of Wedge lemma (Ziegler-Živalievíć 1995)

1. $\widehat{A}(\mathbf{k}) \simeq \bigvee_{q \in Q} \Delta(Q_{<q}) * U_q$ – $\Delta(Q_{<q})$ the order complex "below q", U_q the intersection corresponding to q.
2. $q = (\mathbf{j}_1 \ll \cdots \ll \mathbf{j}_r) \in Q \Rightarrow \Delta(Q_{<q}) \simeq S^{r-2}$ and $U_q = S^{|\mathbf{k}|-r(n-1)}$.
3. q "unordered"$\Rightarrow U_q = *$ – does not contribute !
4. $\widehat{A}(\mathbf{k}) \simeq \bigvee_{q=(\mathbf{j}_1 \ll \cdots \ll \mathbf{j}_r) \in Q} S^{|\mathbf{k}|-r(n-2)-1}$.

Schedules with capacity κ

Replace semaphores of capacity $n-1$ by semaphores with capacity κ. Schedules can be viewed as

- dipaths on κ-skeleton of $\mathbf{R}^n$ (cubified)
- elements in the complement $D^{\kappa+1}(\mathbf{k})$ of $A^{\kappa+1}(\mathbf{k}) =$ $= \{x_{j_1}^{i_1} = \cdots = x_{j_{\kappa+1}}^{i_{\kappa+1}} \mid 1 \leq j_1 < \cdots < j_{\kappa+1} \leq n, 1 \leq i_s \leq k_{j_s}\}$ in $\hat{\mathring{\Delta}}_{\mathbf{k}}$

Strategy

Again use wedge lemma and Alexander duality.
Relevant order complexes: Joins of order complexes of partition complexes – non-singleton parts of size at least $\kappa+1$.
These are homotopy equivalent to wedges of spheres (Björner, Welker; 1995).

Homology and cohomology results

Theorem (Meshulam-R)

1. $\tilde{H}^{|\mathbf{k}|-l-1}(\widehat{A}(\mathbf{k});\mathbf{Z}) = \begin{cases} \mathbf{Z}^{\prod_{i=1}^{n} \binom{k_i}{r}} & l = (n-2)r,\ r > 0 \\ 0 & \textit{otherwise} \end{cases}$
2. $\tilde{H}_l(D(\mathbf{k});\mathbf{Z}) = \begin{cases} \mathbf{Z}^{\prod_{i=1}^{n} \binom{k_i}{r}} & l = (n-2)r,\ r > 0 \\ 0 & \textit{otherwise} \end{cases}$
3. $H_*(D^{\kappa+1}(\mathbf{k});\mathbf{Z})$ *is concentrated in dimensions* $r(\kappa-1),\ r \in \mathbf{Z}_{\geq 0}$.

Configuration spaces and spaces of d-paths

connected by a homotopy equivalence

A sketch

- An element $\mathbf{x} = (x_1, \cdots x_k) \in \mathring{\Delta}_k$ gives rise to a directed piecewise linear path $p_{\mathbf{x}} : I \to [0, k+1]$ with
$$p_{\mathbf{x}}(t) = \begin{cases} 0 & t = 0 \\ i & t = x_i \\ k+1 & t = 1 \end{cases}.$$
- An element $\underline{\mathbf{x}} = (\mathbf{x}_1, \cdots \mathbf{x}_n) \in \prod_1^n \mathring{\Delta}_{k_i} = \mathring{\Delta}_{\mathbf{k}}$ gives rise to a directed piecewise linear path $P_{\underline{\mathbf{x}}} : I \to \mathbf{R}^n$, $P_{\underline{\mathbf{x}}}(t) = (p_{\mathbf{x}_1}(t), \ldots, p_{\mathbf{x}_n}(t))$ from $\mathbf{0}$ to $\mathbf{k}$.
- Only the forbidden configurations in A ((in)-equalities) correspond to dipaths through the forbidden region F (placing the V, P at integers).
- The map $\mathring{\Delta}_{\mathbf{k}} \setminus A \to \vec{P}(\prod_i [0, 2k_i + 1] \setminus F)(\mathbf{0}, 2\mathbf{k} + \mathbf{1}) : \underline{\mathbf{x}} \to P_{\underline{\mathbf{x}}}$ is a homotopy equivalence.

Prodsimplicial vs. configuration space model
A comparison

Dimensions

prodsimplicial model Dimension $\leq l(n-1)$, l the number of "holes" (multiplicative)

configuration space Dimension $\leq 2\sum_i k_i$ (additive)

Questions. Comments

- Can one use the wedge lemma strategy to determine the homotopy type of the complement of the configuration space – in general?
- Determine the stable homotopy type of the configuration space?
- Its homology? Algorithmically?
- Observe: Complicated order complexes!

Thanks

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- the sponsoring centers

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