CHAIN CONTRACTION APPROACH TO (CO)HOMOLOGY COMPUTATION

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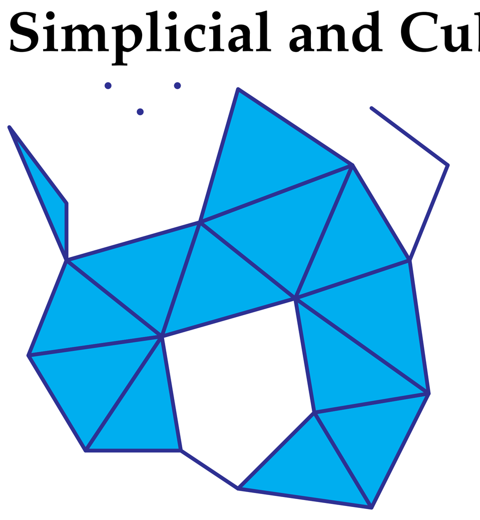
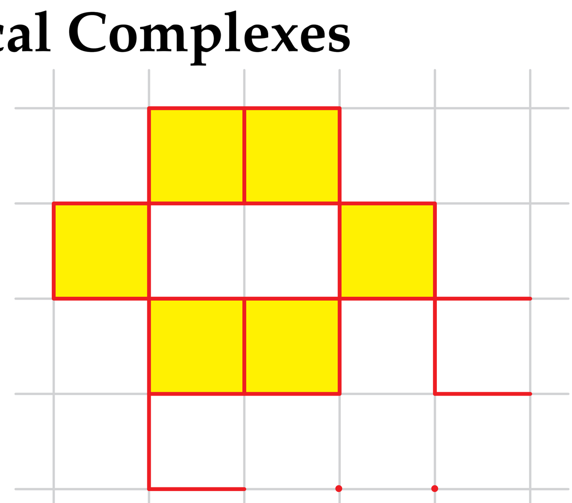
1. INTRODUCTION

Simplicial and Cubical Complexes

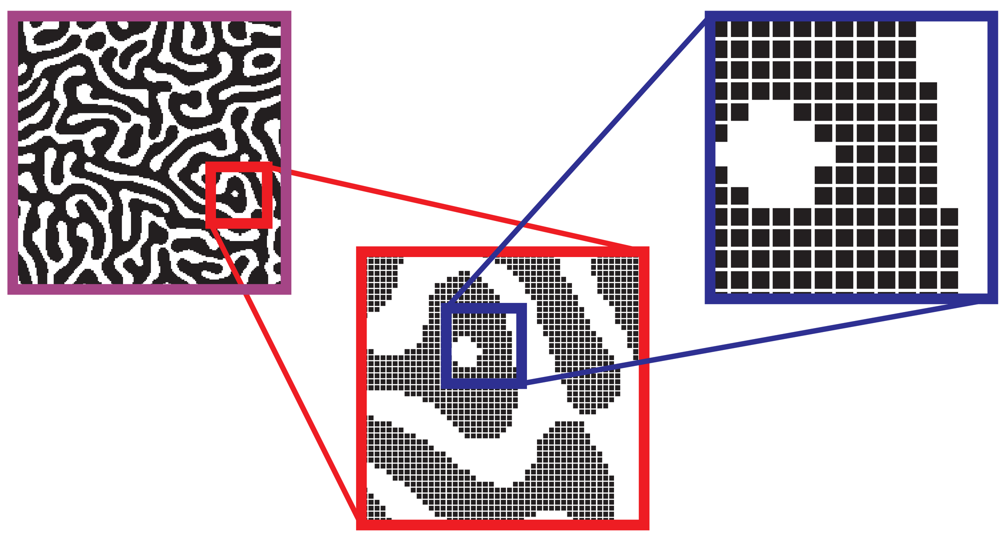


simplices



cubical cells

Cubical complexes appear naturally in many applications. For example, a bitmap image of a pattern consists of square pixels:

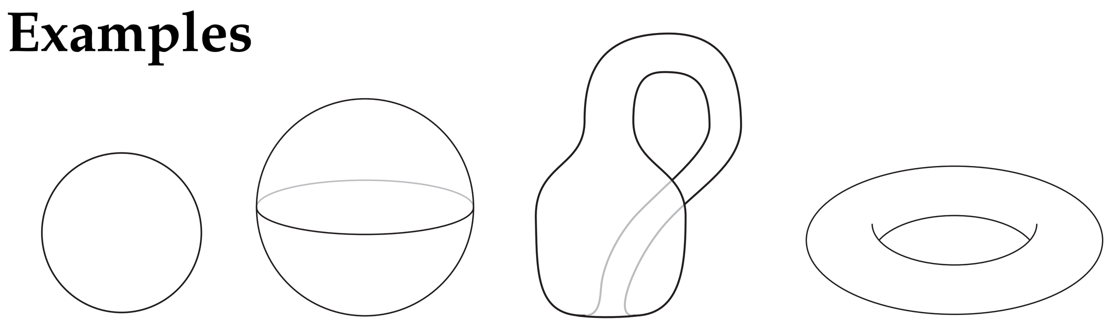


Homology and Cohomology

$C_k :=$ the group of k-dimensional *chains*
$\partial_k\colon C_k \to C_{k-1}$ — the *boundary operator*
$B_k := \operatorname{im} \partial_{k+1} \subset C_k$ — the group of *boundaries*
$Z_k := \ker \partial_k \subset C_k$ — the group of *cycles*
Assume that $\partial \circ \partial \cong 0$. Then $B_k \subset Z_k$.
$H_k := Z_k / B_k$ — the k-th *homology group*

Dual concepts lead to cohomology:
$C^q(K) := \operatorname{Hom}(C_q(K); R)$ — *cochains*
$\big(\delta^q(c)\big)(a) := c\big(\partial_{q+1}(a)\big)$ — *coboundary*
$H^k := Z^k / B^k$ — the k-th *cohomology group*

Examples

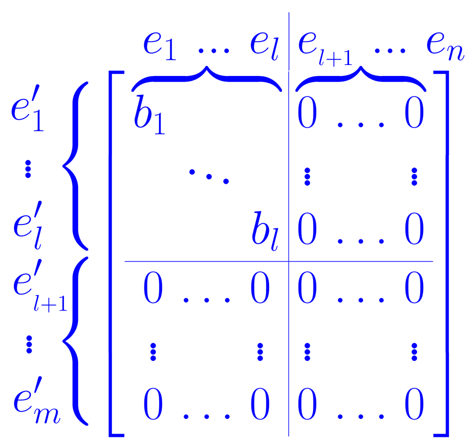


circle	sphere	Klein bottle	torus
$H_0 \cong \mathbb{Z}$	$H_0 \cong \mathbb{Z}$	$H_0 \cong \mathbb{Z}$	$H_0 \cong \mathbb{Z}$
$H_1 \cong \mathbb{Z}$	$H_1 \cong 0$	$H_1 \cong \mathbb{Z}_2 \oplus \mathbb{Z}$	$H_1 \cong \mathbb{Z} \oplus \mathbb{Z}$
$H_2 \cong 0$	$H_2 \cong \mathbb{Z}$	$H_2 \cong 0$	$H_2 \cong \mathbb{Z}$

a double-winding circle map: $f_0 = id_{H_0}$, $f_1(x) = 2x$

Homology Computation via SNF

Change the bases of each C_q to make the matrices D_q of ∂_q in Smith Normal Form:



$b_i \geq 1$; $b_1 \mid b_2 \mid \cdots \mid b_l$

$\{e_{l+1}, \dots, e_n\}$ is a basis for Z_p

$\{e'_1, \dots, e'_l\}$ is a basis for W_{p-1} such that $\partial_p(C_p) \subset W_{p-1}$

$\{b_1 e'_1, \dots, b_l e'_l\}$ = basis for B_{p-1}

$b_i > 1$: torsion of H_{p-1}

Betti numbers:
$\beta_p = \operatorname{rank} Z_p - \operatorname{rank} W_p$

Differential vs. Integral Approach

(a) Differential approach: *reducing* the topological information to a minimal system that describes the degree of connectivity
(b) Integral approach: *representing* the chain contraction of the entire chain complex to a complex that represents its homology

2. MAIN RESULTS

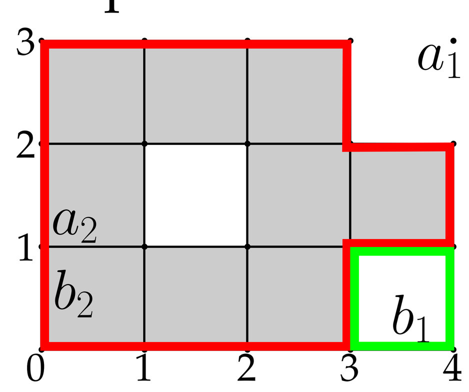
Chain Contraction

A *chain contraction* from C_* to C'_* is a triple (f, g, ϕ) of chain maps:
$f\colon C_* \to C'_*$ (*projection*),
$g\colon C'_* \to C_*$ (*inclusion*),
$\phi\colon C_* \to C_{*+1}$ (*chain homotopy*)
that satisfy the following conditions:
$\mathrm{Id}_C - gf = \partial\phi + \phi\partial$; $\quad fg = \mathrm{Id}_{C'}$;
$f\phi = 0$; $\quad \phi g = 0$; $\quad \phi\phi = 0$.
A *homology integral chain contraction* is a chain contraction in which $\partial\phi\partial = \partial$ and $\phi\partial\phi = \phi$.

Algebraic-Topological Model (AT Model)

An *AT model* of a cell complex K is a homology integral chain contraction from $(C_*(K), \partial)$ to a free chain complex $(M_*, 0)$.

Example:



A basis of M_*:
$a_1 = [4] \times [3]$
$a_2 = [0] \times [1]$
$b_1 = [3, 4] \times [0]$
$b_2 = [0] \times [0, 1]$

Algebraic Minimal Model (AM Model)

An *AM model* of a cell complex K is a chain contraction from $(C_*(K), \partial)$ to a chain complex (M, d) such that each M_q is a free R-module and all the non-zero elements in the SNF of each d_q are non-invertible in R.

Algorithms

AT Model: An incremental algorithm (based on: Delfinado, Edelsbrunner, 1993/1995).
AM Model: Computing the SNF of ∂_q for $q = n, \dots, 1$, and using the change-of-basis matrices to define the chain contraction.

Alexander-Whitney Diagonal (Coproduct)

For a simplex $\sigma = (v_0, \dots, v_n)$:

$$\mathrm{AW}(\sigma) = \sum_{k=0}^{n} \sigma_0^k \otimes \sigma_k^n,$$

$\sigma_i^j = (v_i, \dots, v_j)$, and $\otimes$ is the tensor product.
For a cubical cell $\sigma \in C^n(K)$:

$$\mathrm{AW}(\sigma) = \sum_{J \subset \{1,\dots,n\}} \rho_{J,J'}(\lambda^0_{J'}\sigma \otimes \lambda^1_J \sigma),$$

where: $J' = \{1, \dots, n\} \setminus J$,
$\rho_{J,J'} = (-1)^\nu$, where ν is the number of pairs $(i, j) \in J \times J'$ such that $j < i$,
and $\lambda^m_J \sigma$ is obtained from σ by replacing non-degenerate intervals indexed by J with left ($m = 0$) or right ($m = 1$) endpoints.

Cup Product in Cohomology

Let (f, g, ϕ, M_*, d) be an AM model of a cell complex K, or an AT model if $d = 0$. Then the cup product of two cochains $c\colon M_i \to R$ and $c'\colon M_j \to R$ is the cochain $c \smile c'\colon M_{i+j} \to R$:

$$(c \smile c')(\sigma_{i+j}) = (\mu \circ (c \otimes c') \circ (f \otimes f) \circ \mathrm{AW} \circ g)(\sigma_{i+j}),$$

where $\sigma_{i+j} \in M_{i+j}$, μ is the multiplication in R, and $\otimes$ is the tensor product of homomorphisms: $(h_1 \otimes h_2)(x_1 \otimes x_2) = h_1(x_1) \otimes h_2(x_2)$.

3. SOFTWARE & EXAMPLES

Software Implementation

Generic programming in C++: Templates of algorithms that work for arbitrary cell complexes (simplicial, cubical, etc.) and for arbitrary rings of coefficients ($\mathbb{Z}$, $\mathbb{Z}_p$, etc.)

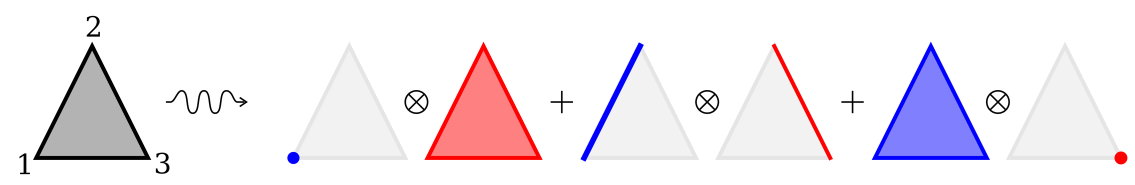
Configurable additional features:
periodic boundary conditions,
reduced (co)homology ($\emptyset \in K$),
relative (co)homology (for pairs).

Programming library interface + executable programs working with text data formats:
computing AT models and AM models,
computing the cohomology ring,
for simplicial and cubical complexes,
with coefficients in $\mathbb{Z}$ and $\mathbb{Z}_p$.

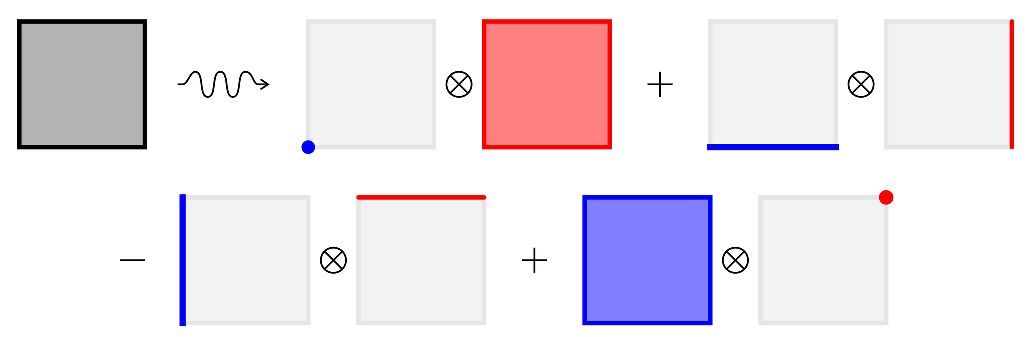
License: GNU GPL version 3.

http://www.pawelpilarczyk.com/chaincon/

Example: 2-dimensional A-W Diagonal

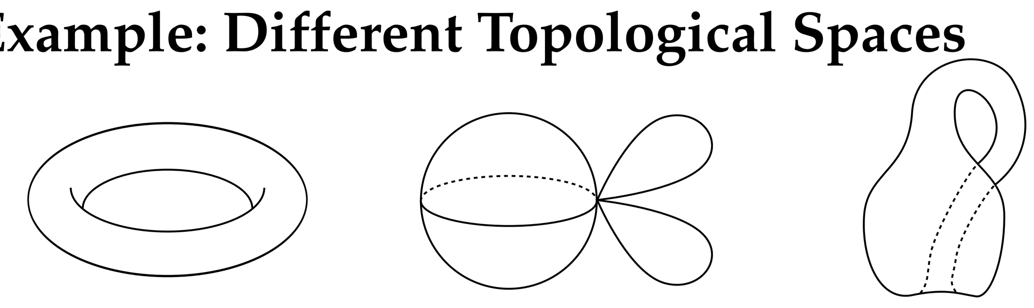


$\mathrm{AW}(1,2,3) =$
$\quad -(1) \otimes (1,2,3) + (1,2) \otimes (2,3) - (1,2,3) \otimes (3)$



$\mathrm{AW}([0,1]\times[0,1]) =$
$\quad [0]\times[0] \otimes [0,1]\times[0,1] + [0,1]\times[0] \otimes [1]\times[0,1] -$
$\quad [0]\times[0,1] \otimes [0,1]\times[1] + [0,1]\times[0,1] \otimes [1]\times[1]$

Example: Different Topological Spaces



All have isomorphic (co)homology over $\mathbb{Z}_2$. Cup product in cohomology distinguishes them. A-W coproduct of 2D homology generators restricted to 1-dimensional chains:
torus: $\mathrm{AW}(c_1) = b_1 \otimes b_2 + b_2 \otimes b_1$;
sphere with two circles: $\mathrm{AW}(c_1) = 0$;
Klein bottle: $\mathrm{AW}(c_1) = b_1 \otimes b_2 + b_2 \otimes b_1 + b_2 \otimes b_2$

Reference

P. Pilarczyk, P. Real, Computation of cubical homology, cohomology, and (co)homological operations via chain contraction, *Adv. Comput. Math.*, DOI: 10.1007/s10444-014-9356-1.

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