# Extremal problems on shadows and hypercuts in simplicial complexes

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## Generalizing graph-theoretic concepts

- Many basic concepts in graph theory have natural counterparts in the realm of high-dimensional simplicial complexes.
- We use the terminology of vertices, edges and faces for simplices of dimension 0,1 and 2 respectively. Let $n$ be the size of the vertex set.
- Identify a face with its corresponding column in the standard $\binom{n}{2} \times \binom{n}{3}$ matrix form of $\partial_2$, the boundary operator over some *underlying field* $\mathbb{F}$.
- A linearly independent set of faces is called *acyclic*.
- A $2$-*hypertree* is a maximal acyclic set of faces. It is straightforward to see that every 2-hypertree is of size $\binom{n-1}{2}$. A 2-*almost-hypertree* is an acyclic set of size $\binom{n-1}{2} - 1$.
- A 2-*hypercut* is a minimal set of faces that intersects with every 2-hypertree.
- The shadow $\mathrm{SH}(S)$ of a set $S$ of faces consists of all the faces $\sigma \notin S$ that are in the $\mathbb{F}$-linear span of $S$.

## Perfect 2-hypercuts over $\mathbb{Q}$

- We construct a perfect 2-hypercut over $\mathbb{Q}$ with $n$ vertices for every prime $n \geq 5$ for which $\mathbb{Z}_n^*$ is generated by $\{-1, 2\}$.
- Let $X = X_n$ be a 2-complex on vertex set $\mathbb{Z}_n$ whose faces are arithmetic progressions of length 3 in $\mathbb{Z}_n$ with difference not in $\{0, \pm 2^{-1}\}$.
- *Theorem: $X$ is an almost-hypertree and $SH(X) = \emptyset$.*
- *Corollary: The complement of $X$ is a perfect 2-hypercut.*
- Proof sketch:
  1. Split the edges and faces by the difference $d \in \mathbb{Z}_n^*$. Namely, Let $E_d = ((a, a+d))_{a=0}^{n-1}$ and $F_d = ((a, a+d, a+2d))_{a=0}^{n-1}$.
  2. Order the rows and columns of $\partial_2(X)$ by $E_1, E_2, E_4, \dots, E_{2^{(n-3)/2}}$ and $F_1, F_2, F_4, \dots, F_{2^{(n-5)/2}}$. $\partial_2(X)$ takes the form

$$\partial_2(X) = \begin{pmatrix} I+Q & 0 & 0 & \dots \\ -I & I+Q^2 & 0 & \dots \\ 0 & -I & \ddots & \dots \\ 0 & 0 & \ddots & I+Q^{2^{\frac{n-1}{2}-2}} \\ 0 & 0 & \dots & -I \end{pmatrix}$$

     Each entry is an $n \times n$ matrix, and $Q$ is the permutation matrix of $b \mapsto b+1 \bmod n$. Indeed,

$$\partial(a, a+d, a+2d) = (a, a+d) + (a+d, a+2d) - (a, a+2d).$$

     ✓ *$X$ is an almost-hypertree.*
  3. Let $u \in \mathbb{Q}^{\binom{n}{2}}$ be defined by $u_e = 2^i$ when $e \in E_{2^i}$. Then for every face $\sigma$, $\langle u, \partial\sigma \rangle = 0 \iff \sigma \in X$.

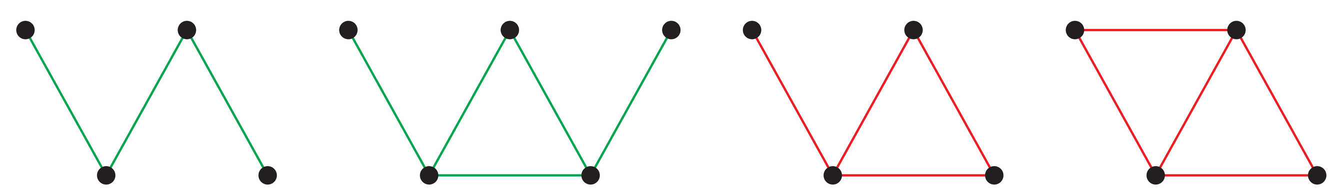
     ✓ *$X$ is shadowless.*

## Main questions

- **Question I: How large can a 2-hypercut be?**
- In graphs, no cut can have more than $\binom{n}{2} - (n-1) + 1$ edges since every cut meets some tree in exactly one edge. However, this bound is far from the correct answer $\lfloor \frac{n^2}{4} \rfloor$.
- Similarly, every 2-hypercut meets some 2-hypertree in exactly one face. Hence, a 2-hypercut has at most $\binom{n}{3} - \binom{n-1}{2} + 1$ faces. A 2-hypercut of this size is called *perfect*.
  **Question II: Do perfect 2-hypercuts exist?**
- In graphs, the least size of the shadow of a forest with two connected components is $\lfloor \frac{n^2}{4} \rfloor$.
  **Question III: How small can the shadow of a 2-almost-hypertree be?**

## Largest 2-hypercuts over $\mathbb{F}_2$

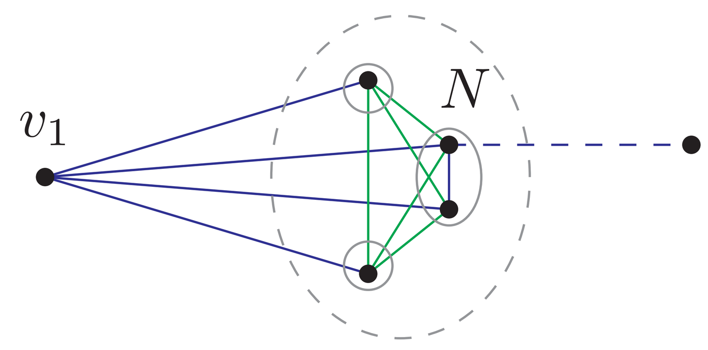
- *Theorem: The largest size of an $n$-vertex 2-hypercut $C$ over $\mathbb{F}_2$ is $|C| \leq \binom{n}{3} - \frac{3}{4} \cdot n^2 + o(n^2)$.*
- Key definition and observations:
  1. A pair of intersecting edges $uv, uv'$ in a graph $G = (V, E)$ are $\Lambda$-*adjacent* if $vv' \notin E$. $G$ is $\Lambda$-*connected* if the $\Lambda$-adjacency relation is connected.



  2. $C$ is a 2-hypercut $\iff$ $C$ is an inclusion-minimal coboundary $\iff$ $\mathrm{link}_v(C)$ is $\Lambda$-connected $\forall v \in V$.
  3. Let $G = \mathrm{link}_v(C)$, $m = |E(\bar{G})|$, $d_1 \geq \dots \geq d_{n-1}$ its degree sequence and $t$ the number of triangles. Then,

$$|\bar{C}| = |1_{\bar{G}} \partial_2| = mn - \sum_i d_i^2 + 4t.$$

- Use the $\Lambda$-connectivity of $G$ to derive constraints on $m, d_1, \dots, d_{n-1}$:
  - $d_1 \leq m/2 + 1$.
  - $d_1 + d_2 \leq (m+n)/2$.
  - $\sum_{i=1}^{k} d_i \leq m - n/2 + O(2^k)$.
- Proof of $d_1 \leq m/2 + 1$. Consider $v_1$ and its neighborhood $N$. *Claim: There are $d_1 - 2$ edges in $\bar{G}$ that meet $N$ and not $v_1$.*
  1. If $N$ has at most 2 connected components in $\bar{G}$ we are done.
  2. Additional components must be connected to $V \setminus N \cup \{v_1\}$:




- Finally, minimizing $|\bar{C}|$ reduces to a quadratic optimization problem in 3 variables $m, d_1, d_2$.
- This can be improved to find a precise bound $|C| \leq \binom{n}{3} - (\frac{3}{4}n^2 - \frac{7}{2}n + 4)$. The bound is tight and attained when $\bar{G}$ is