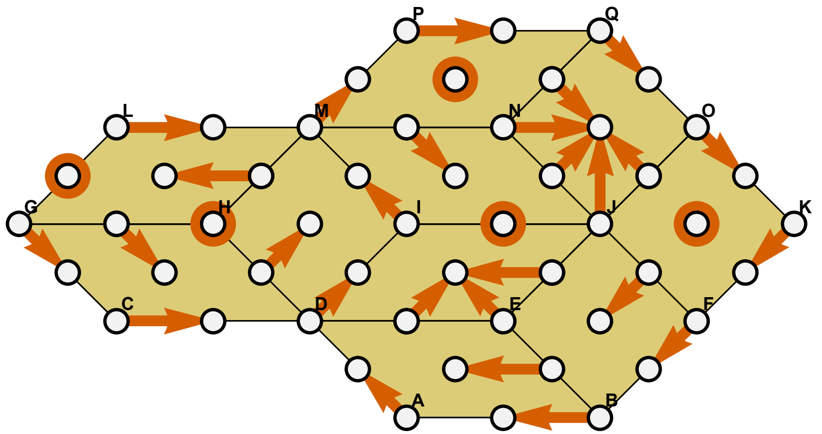
Discrete, Computational and Algebraic Topology

Copenhagen, 12th November 2014

Morse-Forman-Conley theory for combinatorial multivector fields

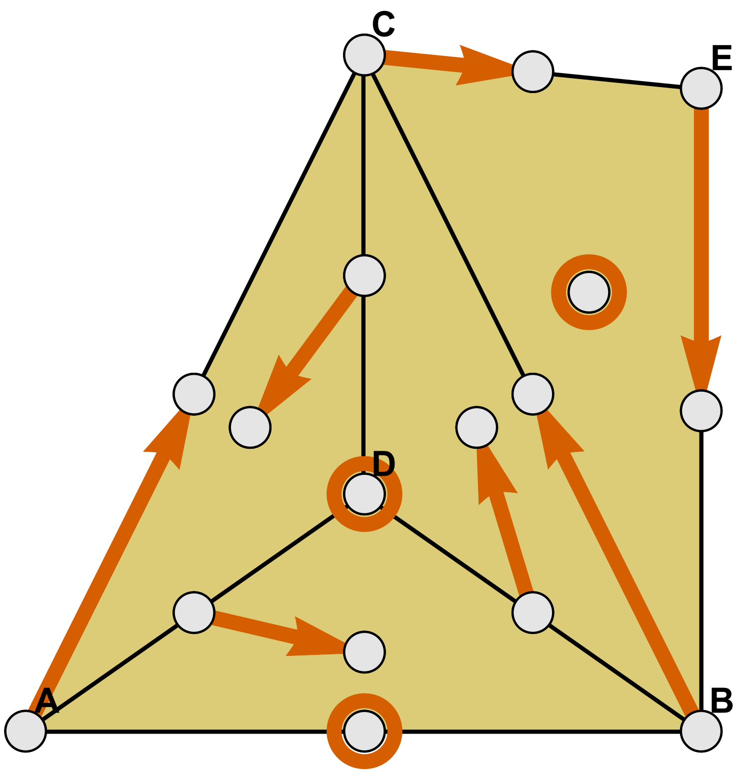
(research in progress)



Marian Mrozek

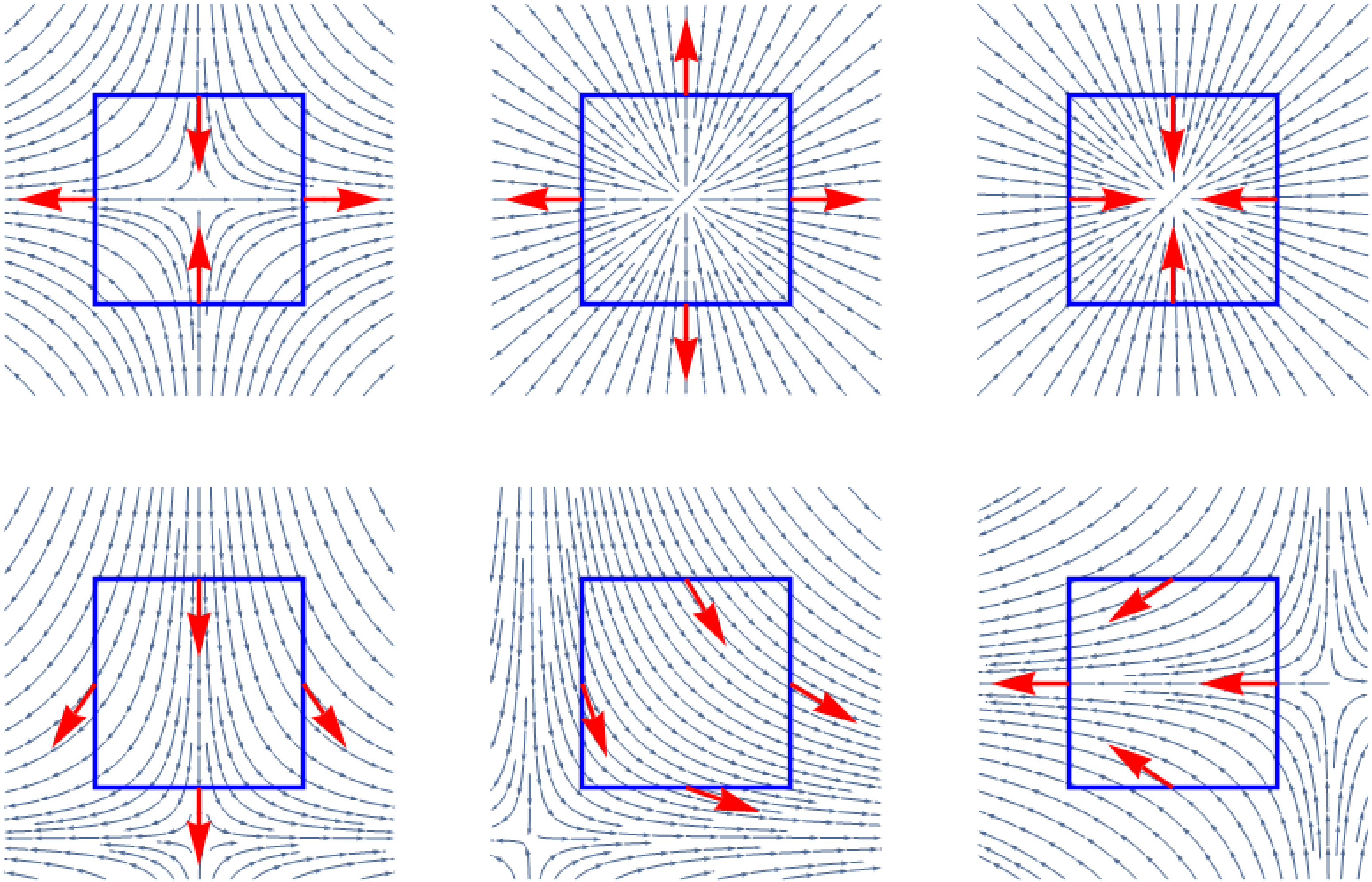
Jagiellonian University, Kraków, Poland

Goals ₂



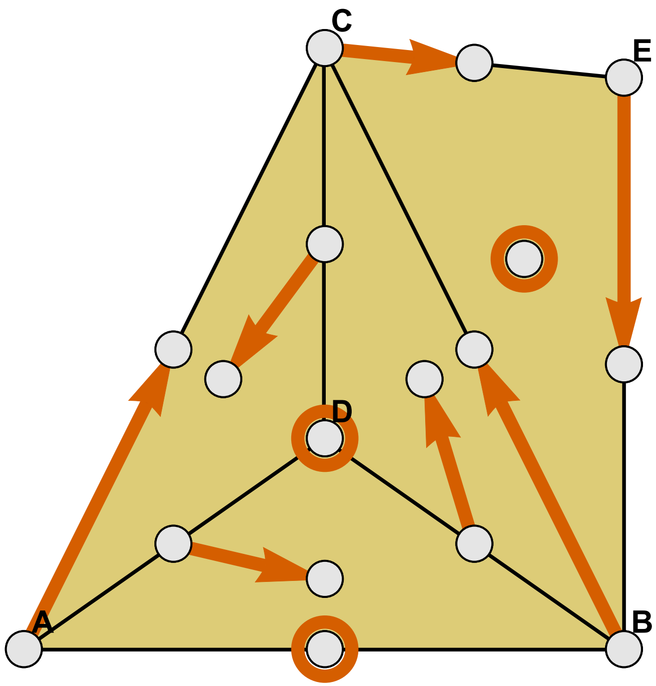
1) Bring Forman's combinatorial vector fields into the framework of classical topological dynamics.

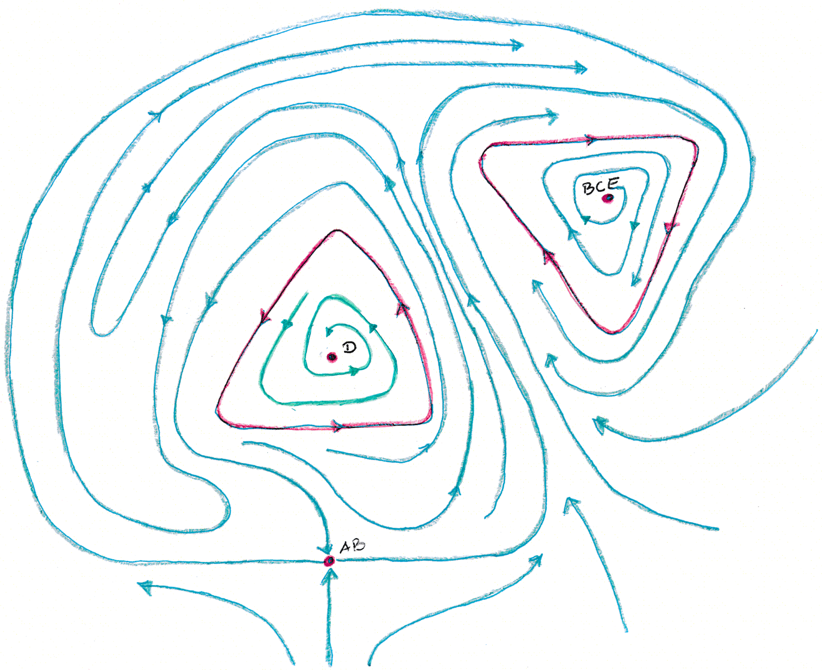
Goals 3



2) Extend the theory to combinatorial multivector fields.

Goals





3) Construct bridges between the combinatorial dynamics on the family of cells of CW complexes and continuous dynamics on the topological space of the complex.

Outline

- Topology of finite sets and Lefschetz complexes
- Combinatorial multivector fields and the associated dynamics
- Isolated invariant sets and the Conley index
- Morse decompositions
- Relation to classical dynamics (joint with T. Kaczynski and Th. Wanner)

Topology of finite sets.

Theorem. (P.S. Alexandroff, 1937) There is a one-to-one correspondence between T_0 topologies and partial orders on a finite set X.

$$\operatorname{cl} A := \{ x \in X \mid \exists_{y \in A}\, x \le y \}$$
$$x \le y \;:\Leftrightarrow\; x \in \operatorname{cl}\{y\}$$

The order complex of a finite poset $(X, \le)$ is the abstract simplicial complex consisting of linearly ordered subsets of X.

Theorem. (M.C. McCord, 1966) Each finite topological space X is weakly homotopy equivalent to its order complex.

Theorem. (J.A. Barmak, 2011) Each h-regular CW complex K is weakly homotopy equivalent to the order complex of the collection of its cells.

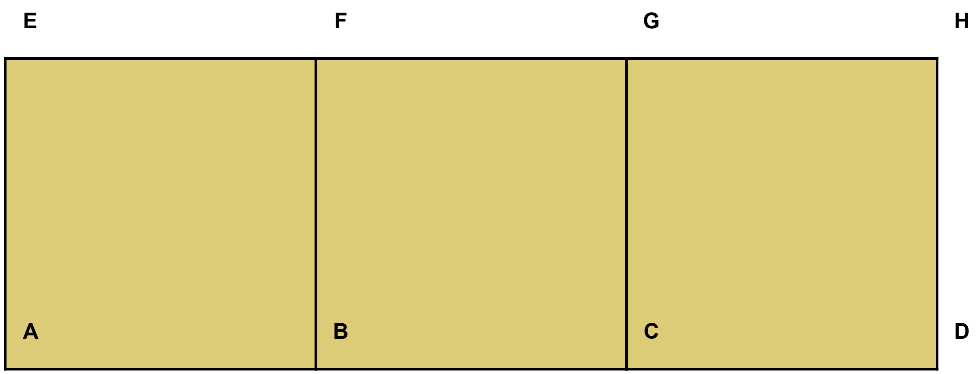
Lefschetz complexes

- $X = (X_q)_{q \in \mathbb{Z}}$ - a finite set with gradation
- $\kappa : X \times X \to R$ s.t. $\kappa(s,t) \neq 0$ implies $s \in X_q$, $t \in X_{q-1}$
- $\partial^\kappa : R(X) \to R(X)$ given by $\partial^\kappa(s) := \sum_{t \in X} \kappa(s,t)t$
- Lefschetz complex - a pair (X, κ) s.t. $(R(X), \partial^\kappa)$ is a free chain complex.
- $H^\kappa(X)$ - homology of $(R(X), \partial^\kappa)$
- $t \prec_\kappa s :\Leftrightarrow \ \kappa(s,t) \neq 0$ induces $\leq_\kappa$ - the Lefschetz partial order
- T_0 Lefschetz topology on X - given by $\leq_\kappa$

Homology of Lefschetz complexes

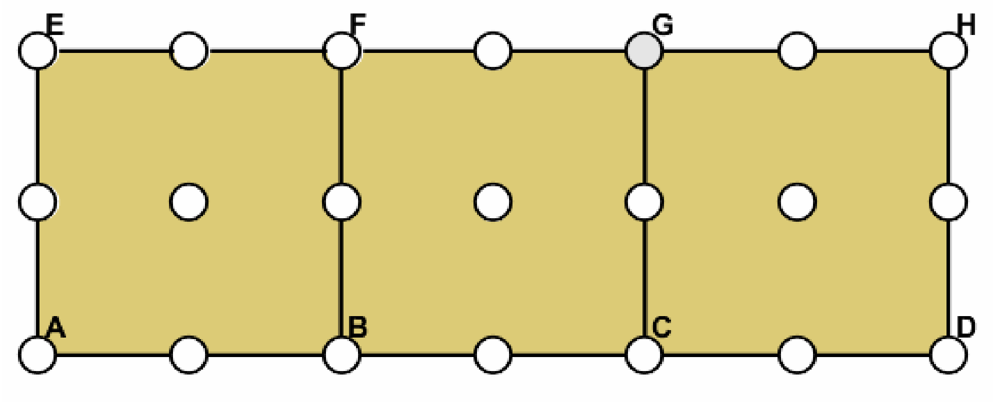
$H^{\kappa}(X) \cong H(X)$ need not be true. It is true if X is the collection of cells of a CW complex.

Homology of Lefschetz complexes

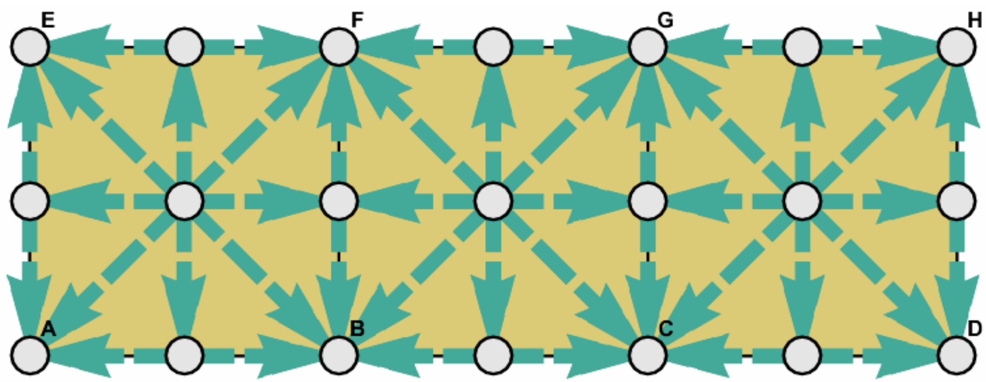


$H^{\kappa}(X) \cong H(X)$ need not be true. It is true if X is the collection of cells of a CW complex.

Homology of Lefschetz complexes



$H^{\kappa}(X) \cong H(X)$ need not be true. It is true if X is the collection of cells of a CW complex.



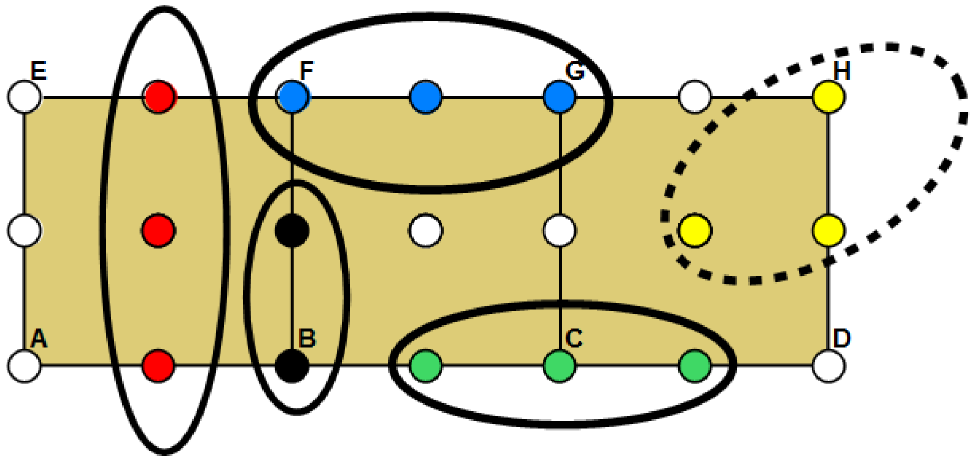
$H^{\kappa}(X) \cong H(X)$ need not be true. It is true if X is the collection of cells of a CW complex.

κ-subcomplexes

- $A \subset X$ is a κ-subcomplex of X if $(A, \kappa_{|A\times A})$ is a Lefschetz complex.
- $\operatorname{mo} A := \operatorname{cl} A \setminus A$ - mouth of A
- A is proper if $\operatorname{mo} A$ is closed.

Proposition. Every proper subset of X is a κ-subcomplex. In particular open and closed sets (in the Lefschetz topology) are κ-subcomplexes.

κ-subcomplexes

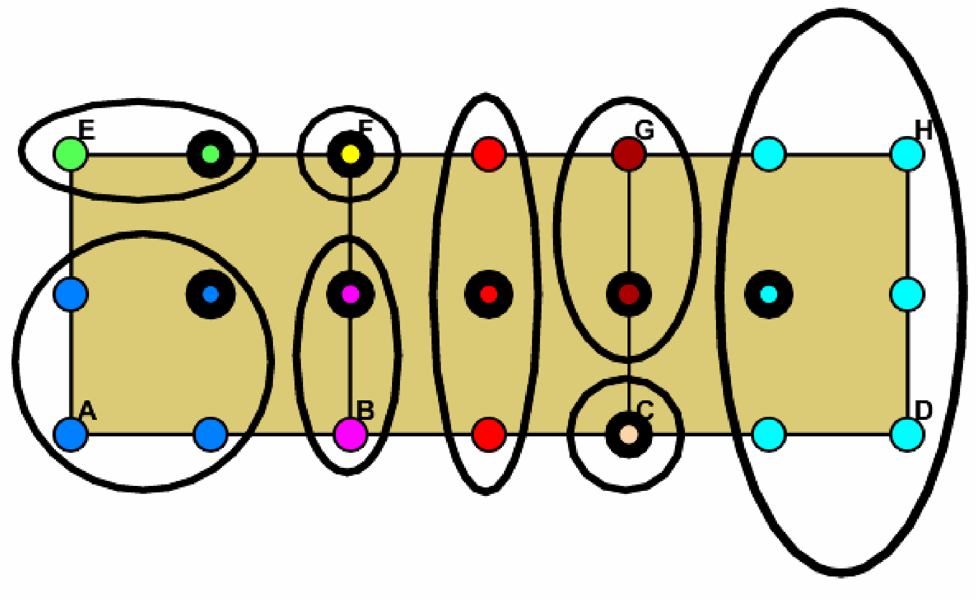


- $A \subset X$ is a κ-subcomplex of X if $(A, \kappa_{|A\times A})$ is a Lefschetz complex.
- $\operatorname{mo} A := \operatorname{cl} A \setminus A$ - mouth of A
- A is proper if $\operatorname{mo} A$ is closed.

Proposition. Every proper subset of X is a κ-subcomplex. In particular open and closed sets (in the Lefschetz topology) are subcomplexes.

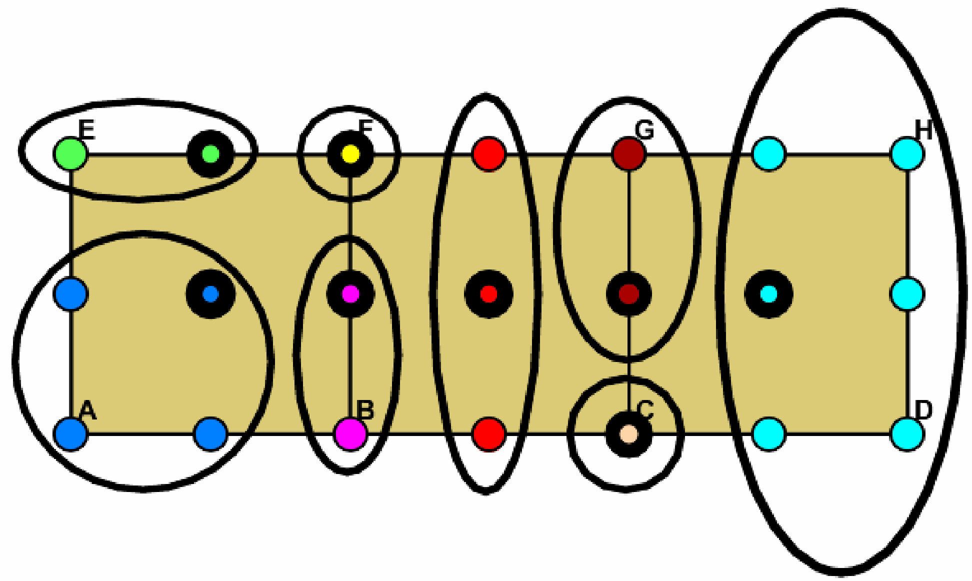
A proper $A \subset X$ is a zero space if $H^\kappa(A) = 0$.

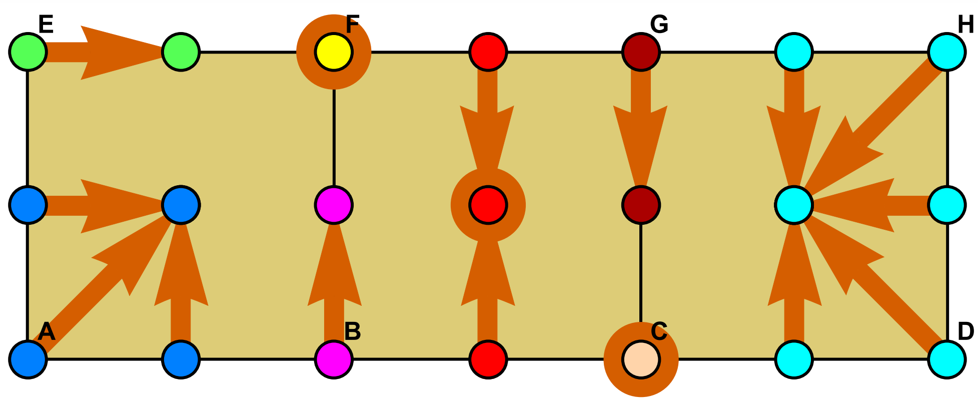
Combinatorial multivector fields



- A multivector is a proper $V \subset X$ with a unique maximal element.
- A multivector field is a partition $\mathcal{V}$ of X into multivectors.
- V is regular if V is a zero space. Otherwise it is critical.

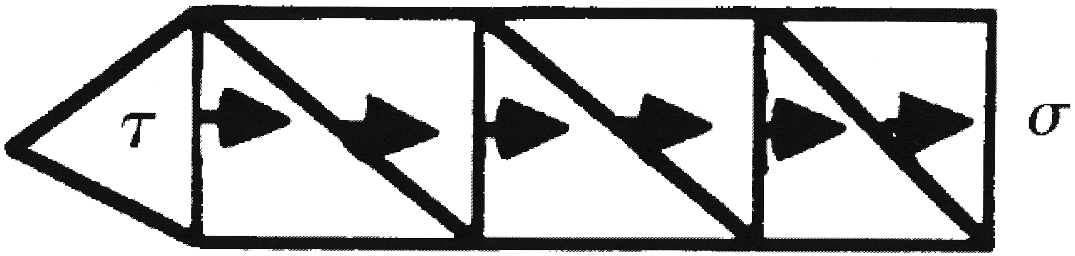
Combinatorial multivector fields





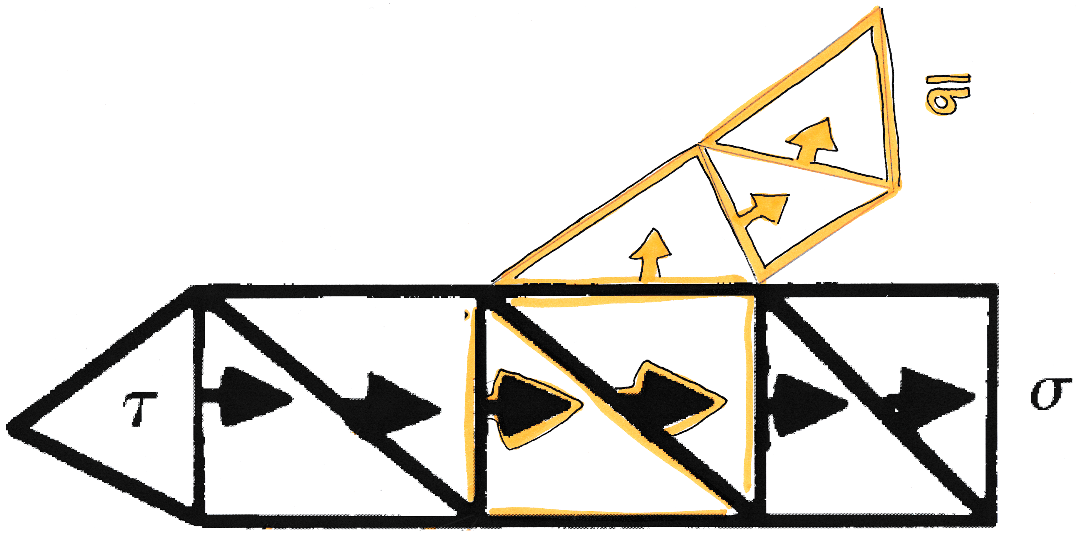
- A multivector is a proper $V \subset X$ with a unique maximal element.
- A multivector field is a partition $\mathcal{V}$ of X into multivectors.
- V is regular if V is a zero space. Otherwise it is critical.

Forman's paths



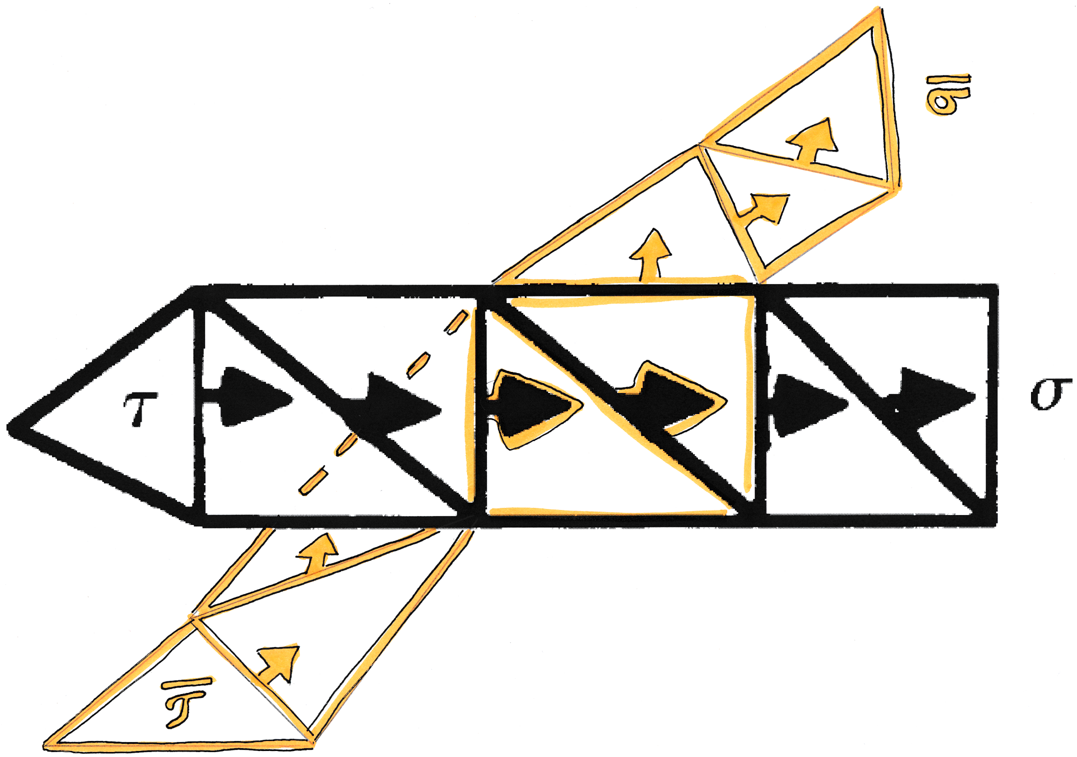
- Forman's path:
  a sequence of $\mathcal{V}$-arrows $x = (x_i^-, x_i^+)_{i=1}^n$ such that $x_i^+ \succ x_{i+1}^-$
- A path is closed if $x_n^+ = x_1^-$

Forman's paths



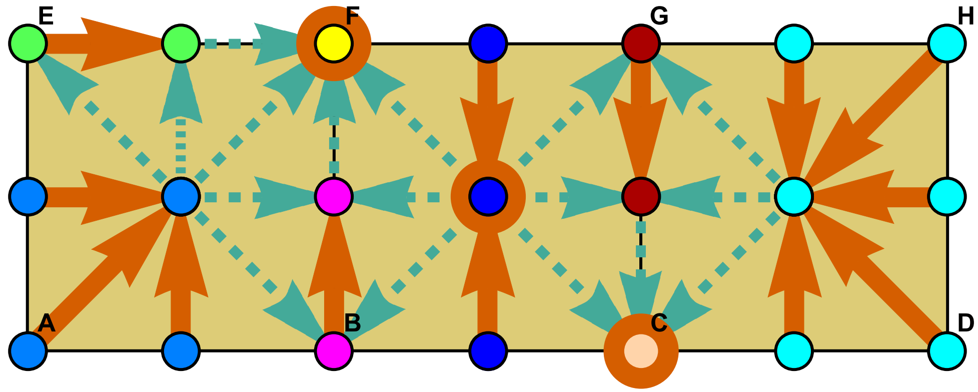
- Forman's path:
  a sequence of $\mathcal{V}$-arrows $x = (x_i^-, x_i^+)_{i=1}^n$ such that $x_i^+ \succ x_{i+1}^-$
- A path is closed if $x_n^+ = x_1^-$

Forman's paths 18



- Forman's path:
  a sequence of $\mathcal{V}$-arrows $x = (x_i^-, x_i^+)_{i=1}^n$ such that $x_i^+ \succ x_{i+1}^-$
- A path is closed if $x_n^+ = x_1^-$

Combinatorial multivalued map



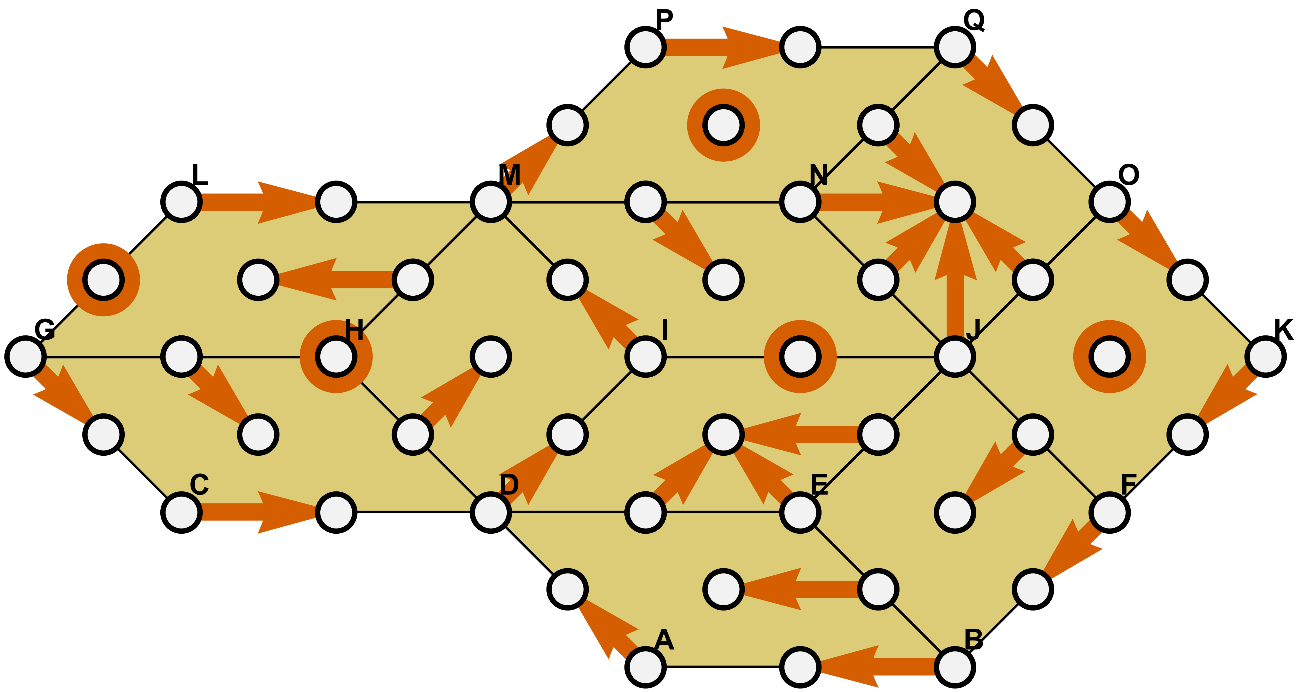
$$x^+ := \max[x]_{\mathcal{V}}$$

$$\langle x\rangle_{\mathcal{V}} := \begin{cases} [x]_{\mathcal{V}} & \text{if } [x]_{\mathcal{V}} \text{ is regular,} \\ [x]_{\mathcal{V}} \setminus \{x^+\} & \text{otherwise.} \end{cases}$$

Definition. The multivalued map $\Pi_{\mathcal{V}} : X \rightrightarrows X$ given by

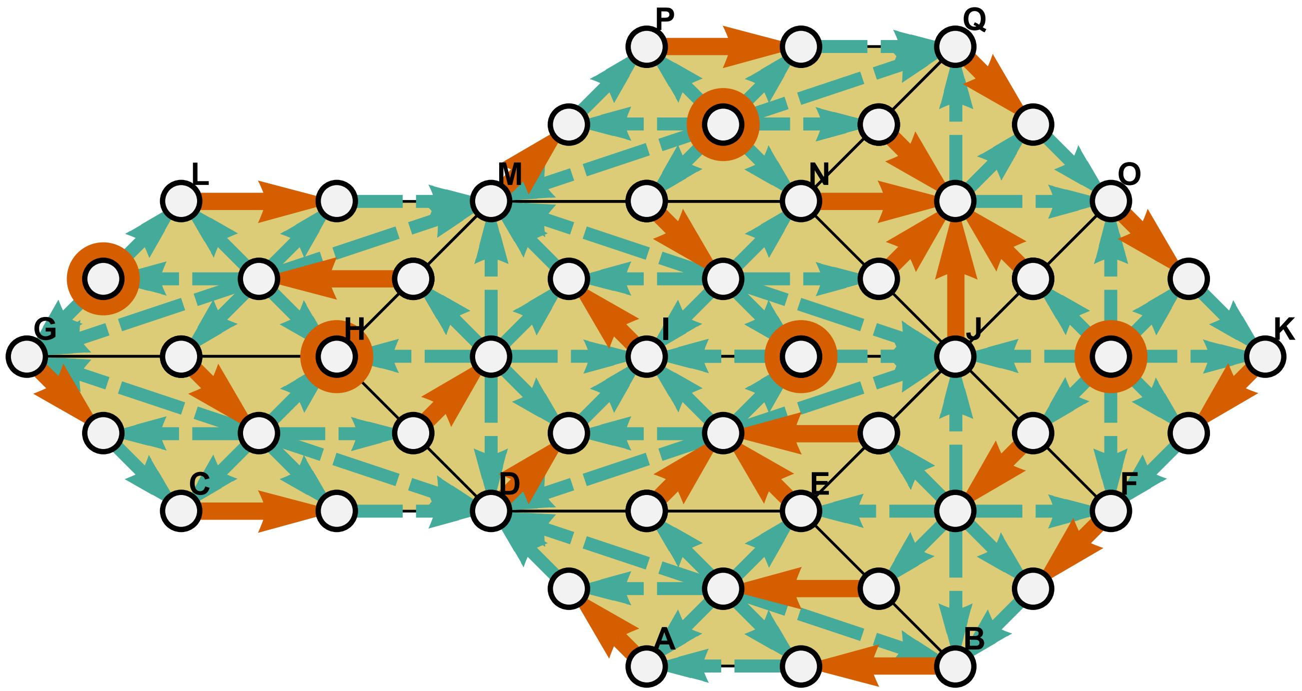
$$\Pi_{\mathcal{V}}(x) = \begin{cases} \{x^+\} & \text{if } x < x^+, \\ \operatorname{cl} x \setminus \langle x\rangle_{\mathcal{V}} & \text{otherwise.} \end{cases}$$

is the combinatorial multivalued map associated with $\mathcal{V}$.



- A partial map $\gamma : \mathbb{Z} \multimap X$ is a solution of $\mathcal{V}$ if

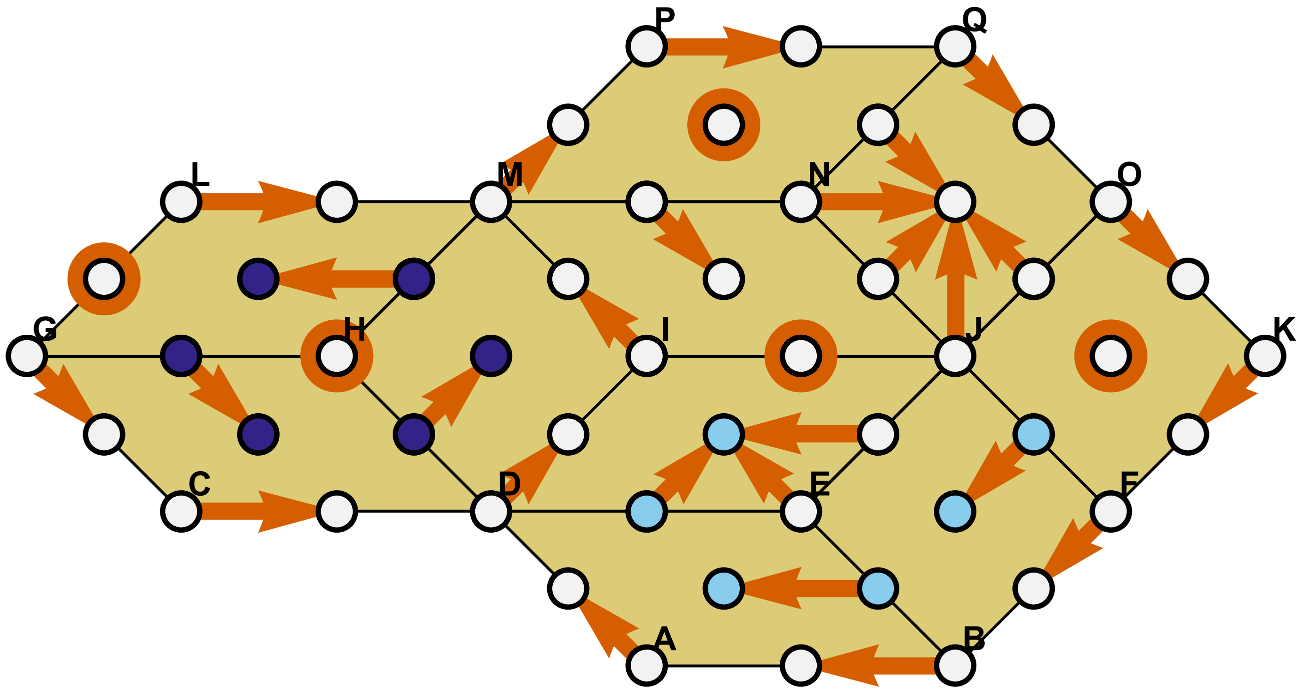
$$\gamma(i+1) \in \Pi_{\mathcal{V}}(\gamma(i)) \text{ for } i, i+1 \in \operatorname{dom} \gamma.$$



- A partial map $\gamma : \mathbb{Z} \multimap\!\!\to X$ is a solution of $\mathcal{V}$ if

$$\gamma(i+1) \in \Pi_{\mathcal{V}}(\gamma(i)) \text{ for } i, i+1 \in \operatorname{dom} \gamma.$$

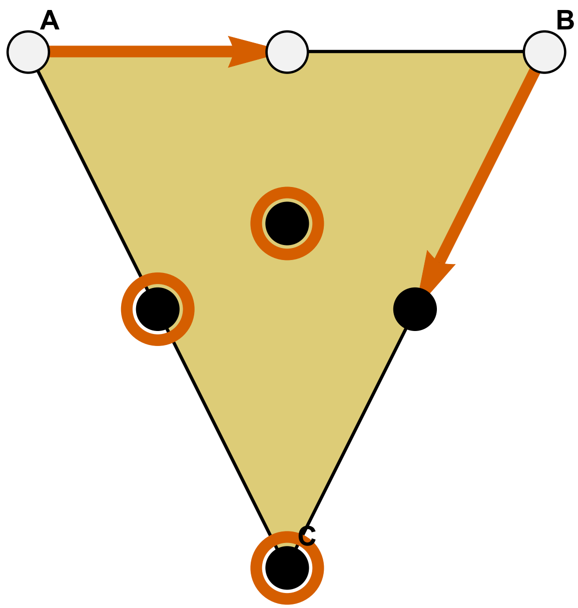
Solutions and paths



- A partial map $\gamma : \mathbb{Z} \multimap X$ is a solution of $\mathcal{V}$ if

$$\gamma(i+1) \in \Pi_{\mathcal{V}}(\gamma(i)) \text{ for } i, i+1 \in \operatorname{dom} \gamma.$$

Invariant sets and $\mathcal{V}$-compatibility



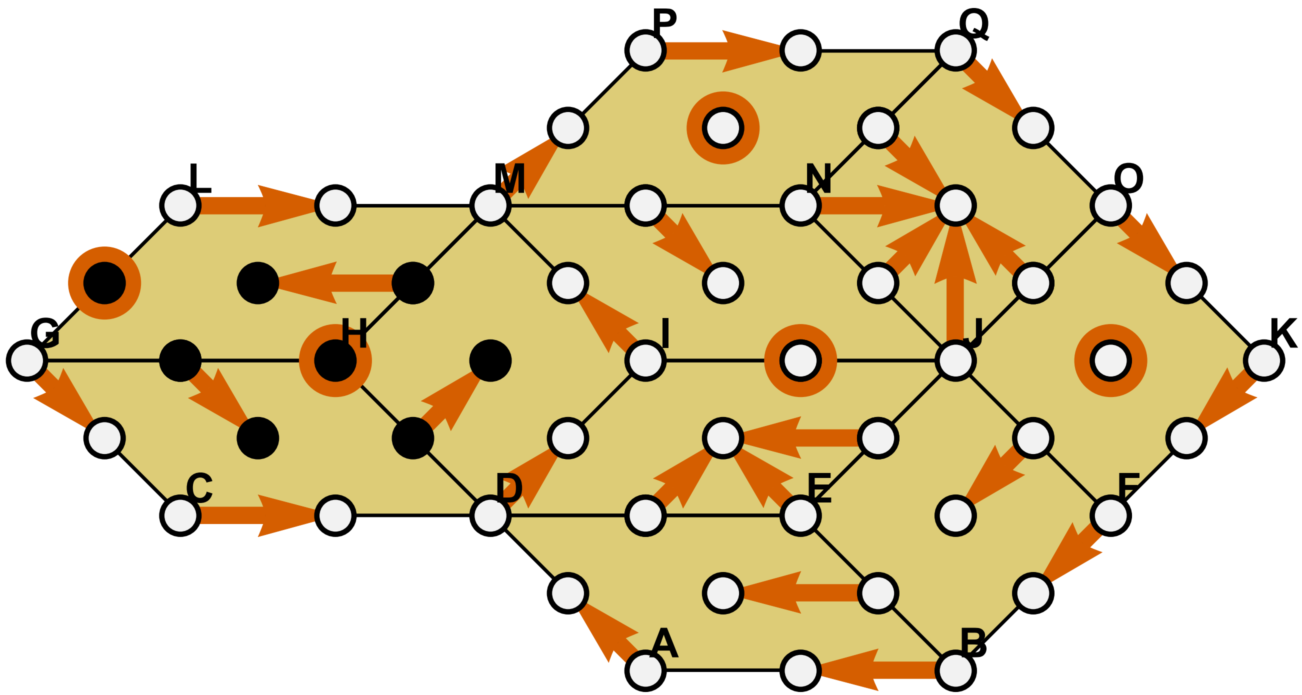
$$\operatorname{Inv} A := \{\, x \in A \mid \exists\, \varrho : \mathbb{Z} \to A \text{ a solution s.t. } \varrho(0) = x.\,\}$$

Let $S \subset X$.

Definition. S is $\mathcal{V}$-invariant if $\operatorname{Inv} S = S$.

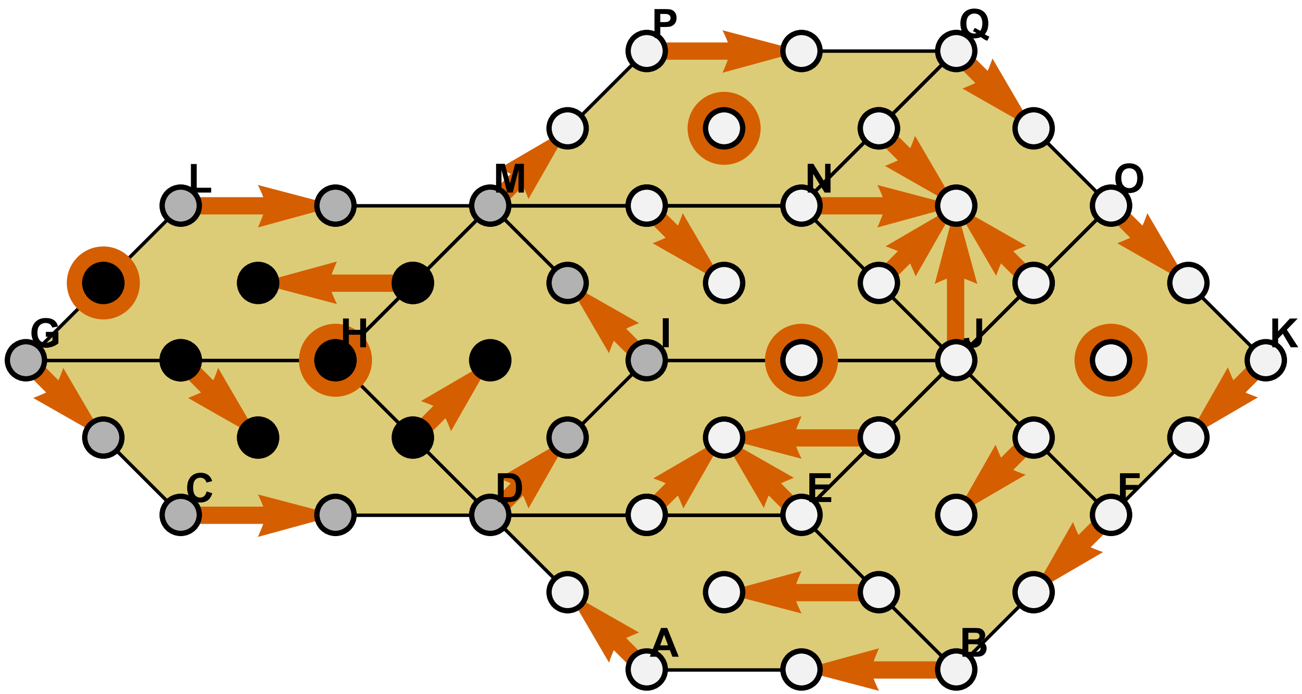
Definition. $A \subset X$ is $\mathcal{V}$-compatible iff

$$x \in A \;\Rightarrow\; [x]_{\mathcal{V}} \subset A$$



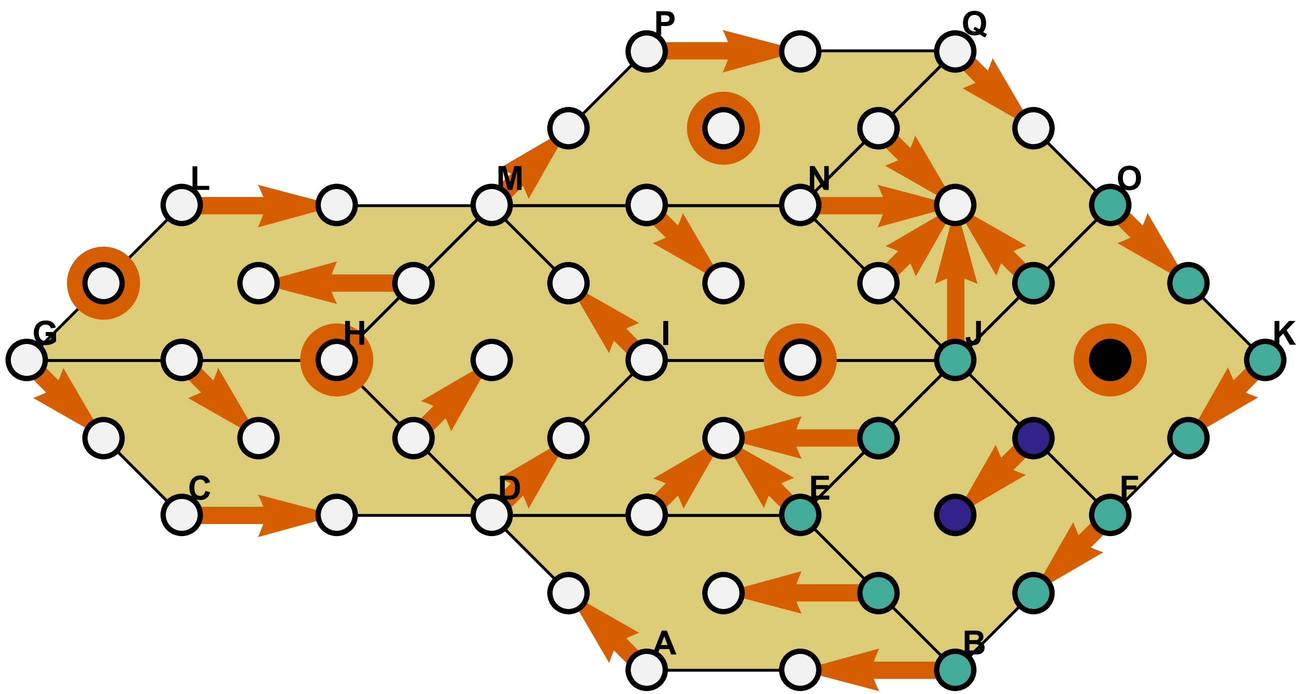
Definition. S is an isolated invariant set of $\mathcal{V}$ if it is invariant, $\mathcal{V}$-compatible and $\operatorname{mo} S$ is closed.

Isolated invariant sets



Definition. S is an isolated invariant set of $\mathcal{V}$ if it is invariant, $\mathcal{V}$-compatible and $\operatorname{mo} S$ is closed.

Index pairs



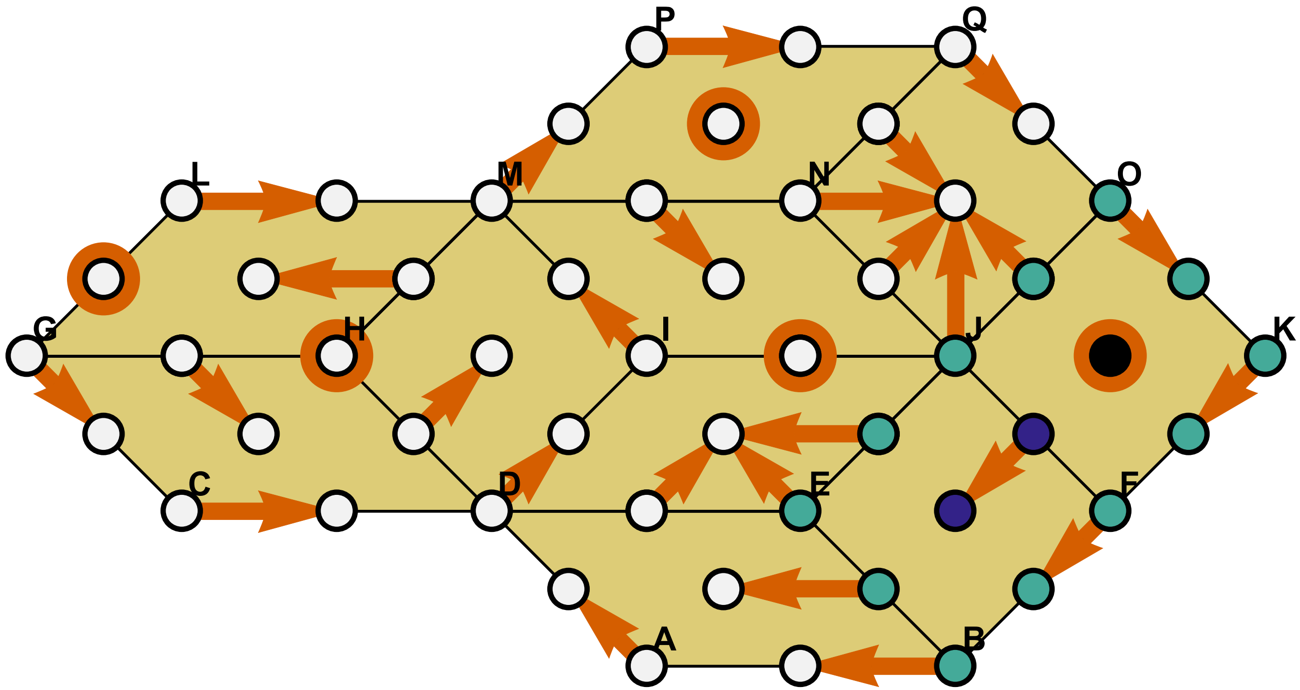
Definition. A pair $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2)$ of closed subsets of $\mathcal{X}$ is an index pair for $\mathcal{S}$ iff

(i) $x \in \mathcal{P}_2,\ y \in \mathcal{P}_1 \cap \Pi_{\mathcal{V}}(x) \ \Rightarrow\ y \in \mathcal{P}_2$,

(ii) $x \in \mathcal{P}_1,\ \Pi_{\mathcal{V}}(x) \setminus \mathcal{P}_1 \neq \emptyset \ \Rightarrow\ x \in \mathcal{P}_2$,

(iii) $\mathcal{S} = \operatorname{Inv}(\mathcal{P}_1 \setminus \mathcal{P}_2)$.

Conley index



Theorem.

- For every S an isolated invariant set $(\operatorname{cl} S, \operatorname{mo} S)$ is an index pair for S.
- If P and Q are index pairs for S, then $H^\kappa(P_1 \setminus P_2)$ and $H^\kappa(Q_1 \setminus Q_2)$ are isomorphic .

Conley index

Definition. The Conley index of S is the homology

$$H^{\kappa}(P_1 \setminus P_2) = H^{\kappa}(P_1, P_2)$$

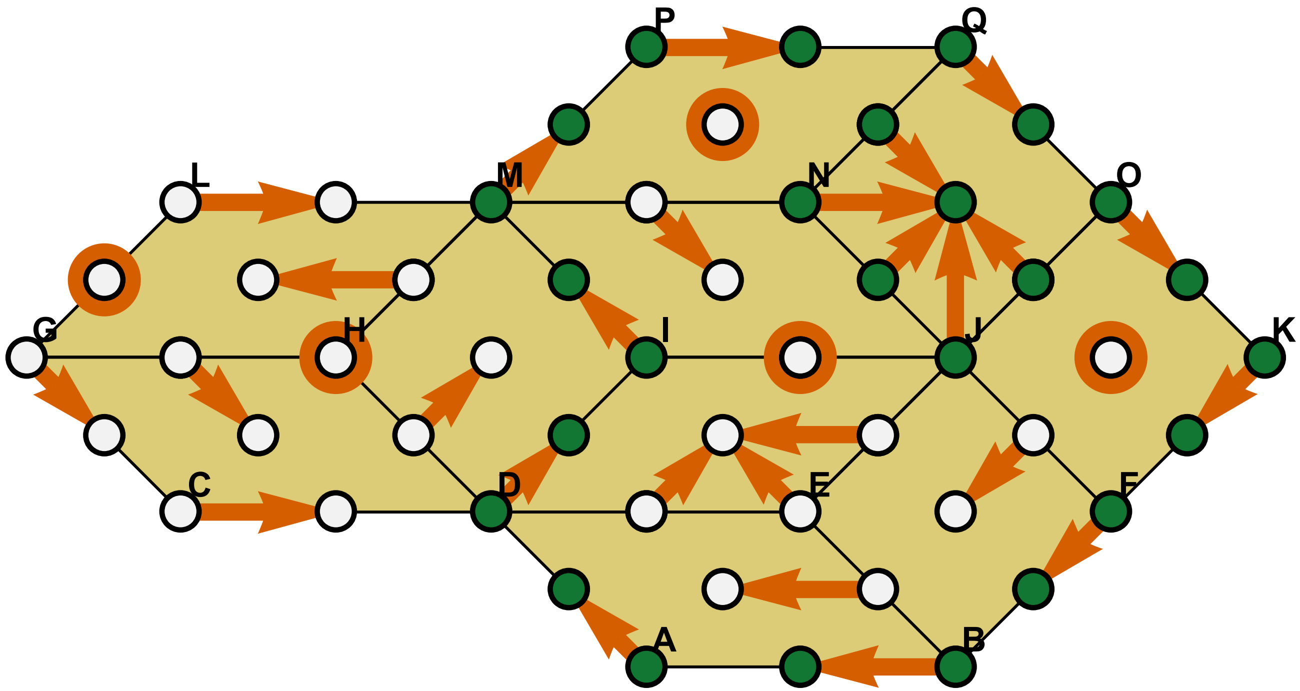
for any index pair P of S.

The Conley polynomial of S is

$$p_S(t) := \sum_{i=0}^{\infty} \beta_i(S) t^i,$$

where $\beta_i(S) :=$ rank $H_i(P_1, P_2)$.

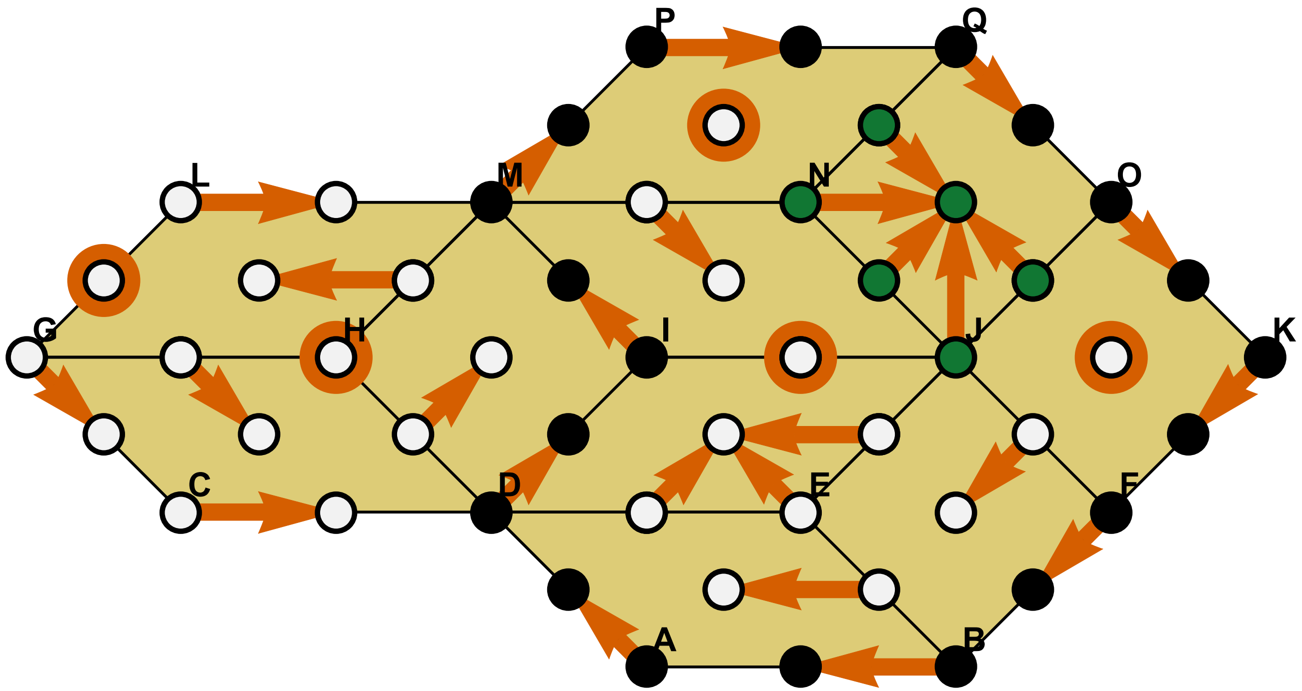
Attractors



Let $S \subset X$ be isolated invariant.

- $N \subset S$ is a trapping region if for every solution $\gamma : \mathbb{Z}^+ \to S$ condition $\gamma(0) \in N$ implies $\operatorname{im} \gamma \subset N$.
- A is an attractor in S if iff there is a trapping region N such that $A = \operatorname{Inv} N$.

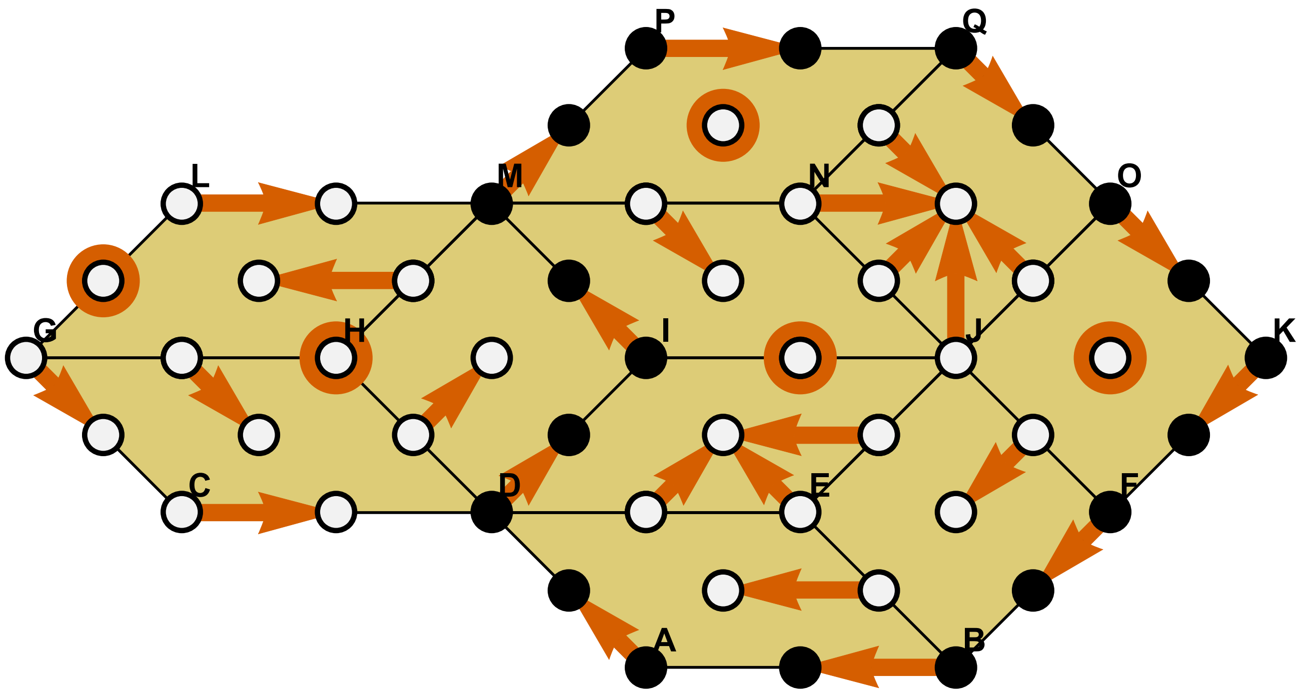
Attractors



Let $S \subset X$ be isolated invariant.

- $N \subset S$ is a trapping region if for every solution $\gamma : \mathbb{Z}^+ \to S$ condition $\gamma(0) \in N$ implies $\operatorname{im} \gamma \subset N$.
- A is an attractor in S if iff there is a trapping region N such that $A = \operatorname{Inv} N$.

Attractors



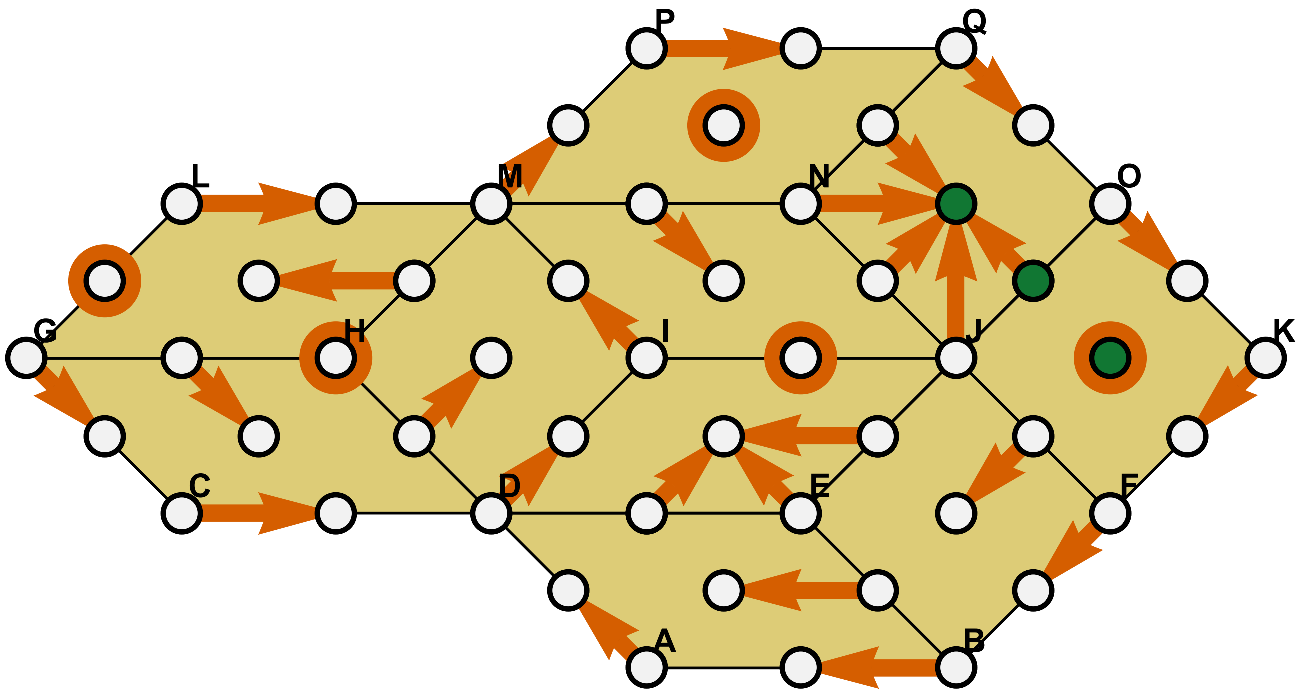
Theorem. The following conditions are equivalent:

(i) A is an attractor,

(ii) A is invariant and closed in S,

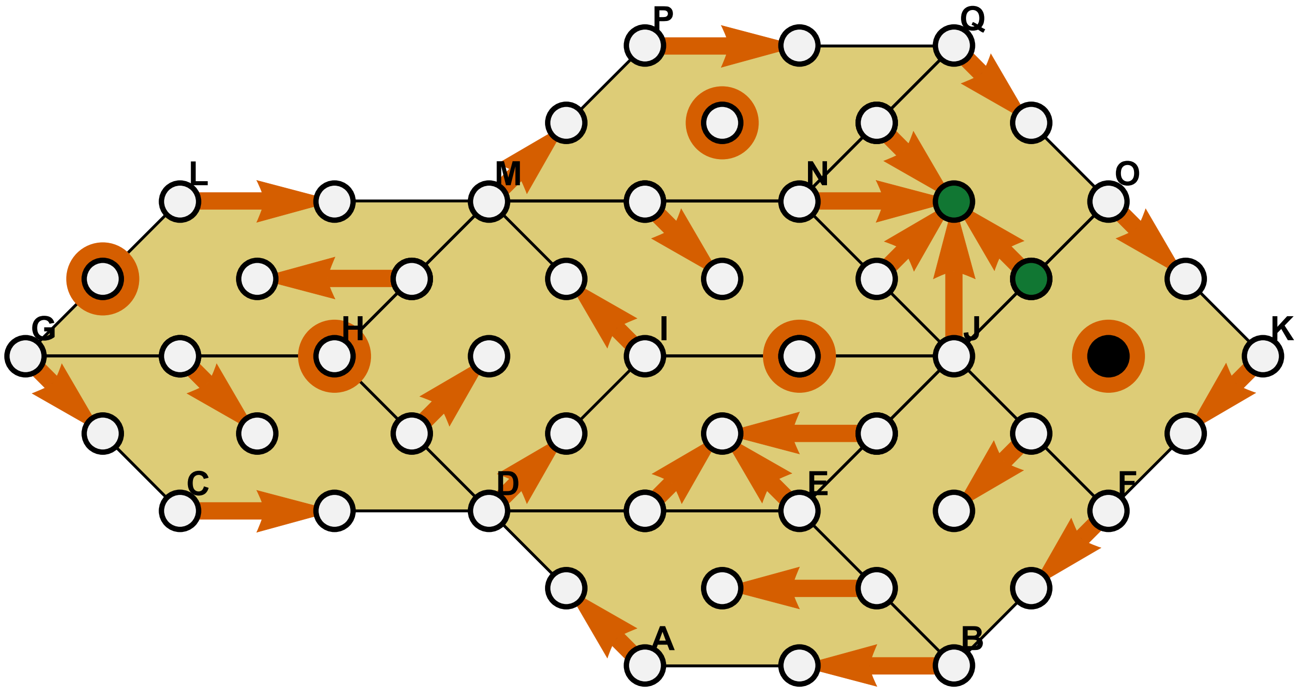
(iii) A is isolated invariant and closed in S.

Repellers



- $N \subset S$ is a backward trapping region if N is $\mathcal{V}$-compatible and for every solution $\gamma : \mathbb{Z}^- \to S$ condition $\gamma(0) \in N$ implies $\operatorname{im} \gamma \subset N$.
- R is an repeller in S if iff there is a backward trapping region N such that $R = \operatorname{Inv} N$.

Repellers

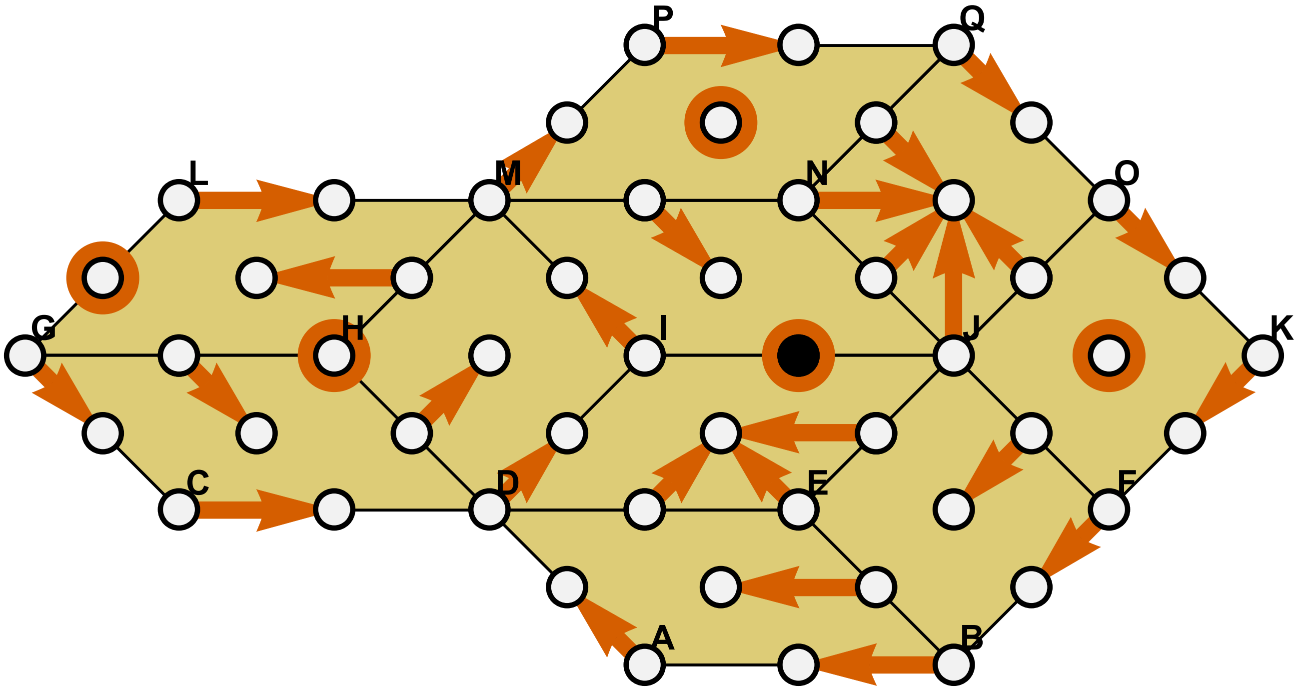


Theorem. The following conditions are equivalent:

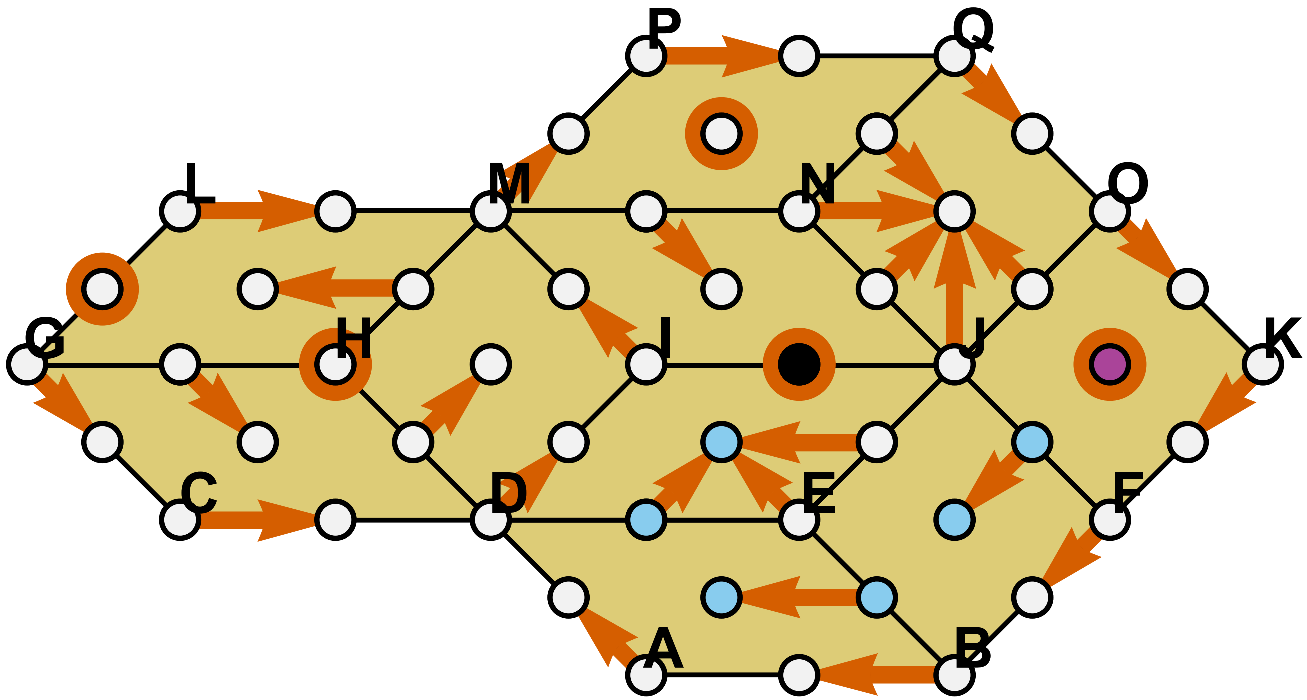
(i) R is a repeller,

(ii) R is isolated invariant and open in S.

Hyperbolic fixed point



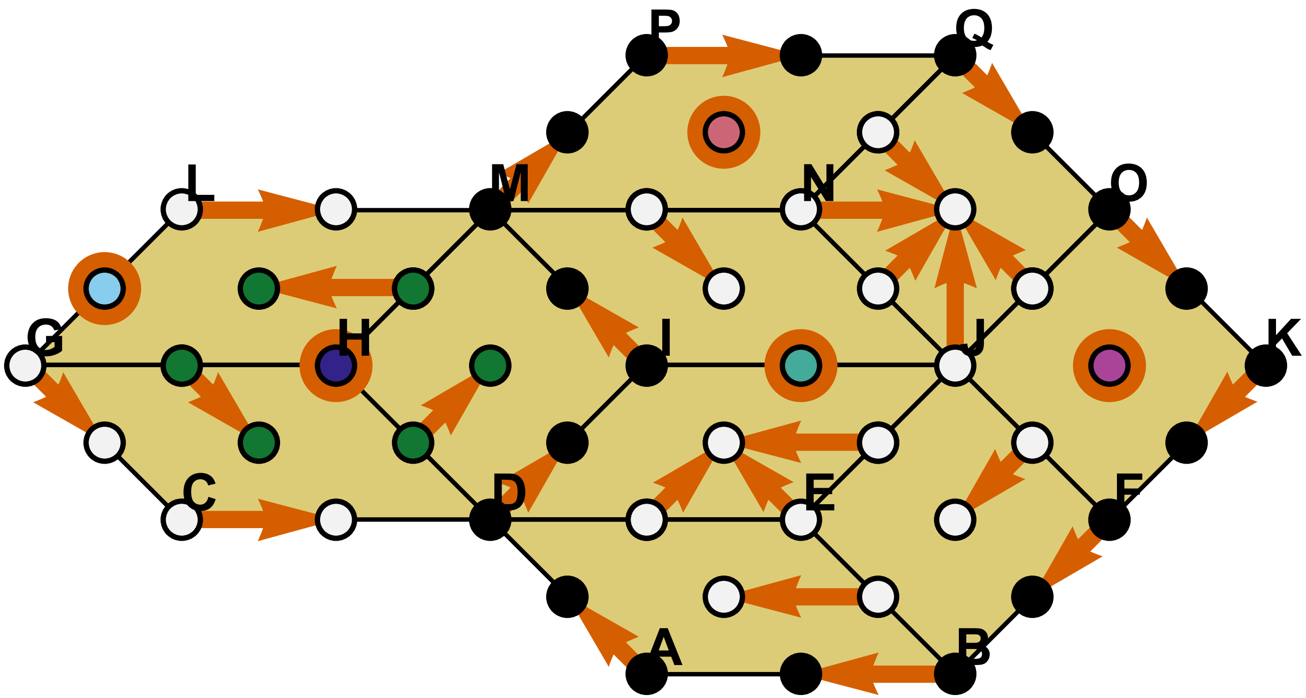
α and ω limit sets



$\varrho : \mathbb{Z} \to S$ - a full solution. The α and ω limit sets of ϱ are

$$\alpha(\varrho) := \operatorname{Inv} \operatorname{im} \varrho_{\mathbb{Z}^-},$$
$$\omega(\varrho) := \operatorname{Inv} \operatorname{im} \varrho_{\mathbb{Z}^+}.$$

Morse decompositions

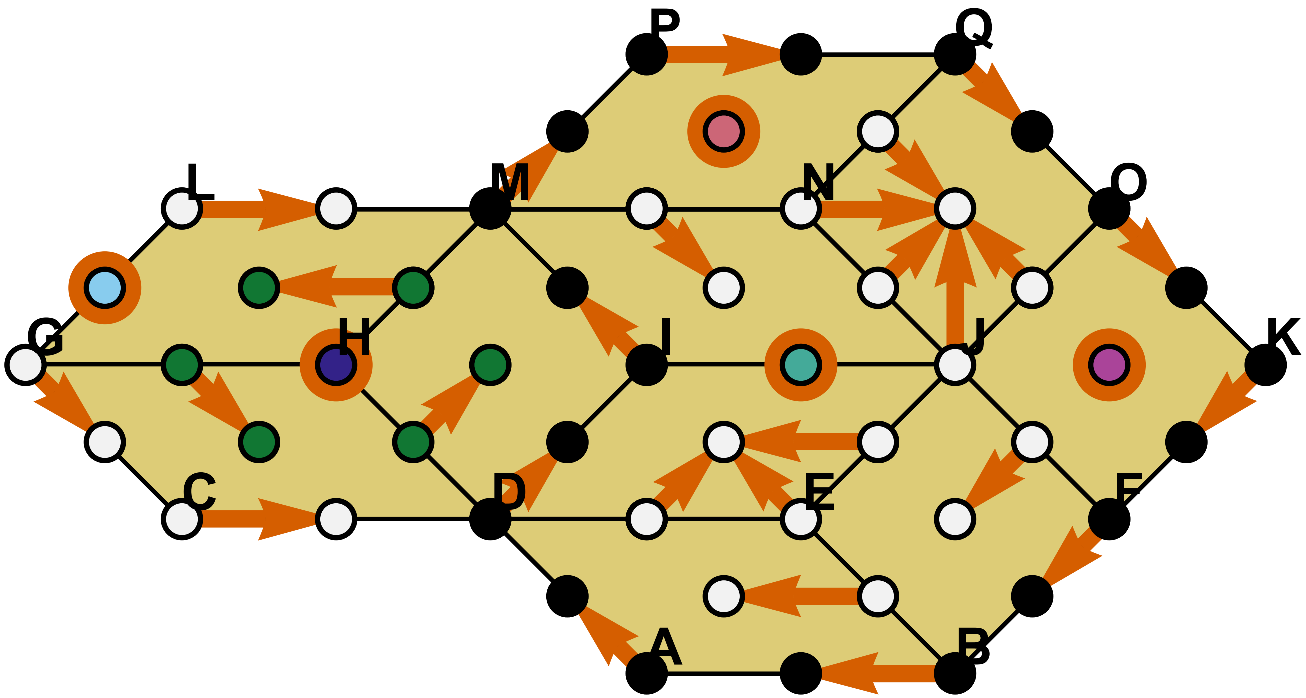


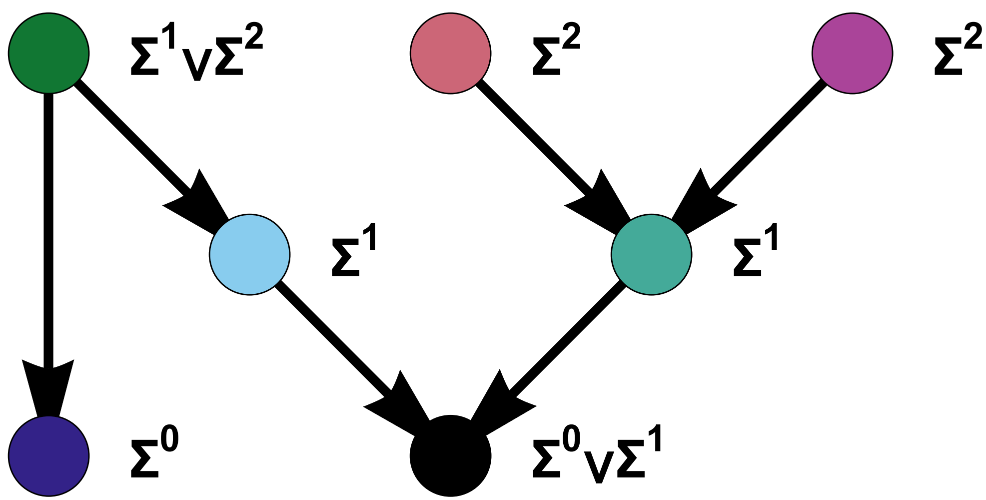
Definition. The collection $M = \{ M_p \mid p \in P \}$ is a Morse decomposition of S if M is a family of mutually disjoint invariant subsets of S and for every solution ϱ such that

$$\varrho(0) \in S \setminus \bigcup_{p \in P} M_p$$

there exists $p, p' \in P$ such that $p < p'$, $\alpha(\varrho) \subset S_1$, $\omega(\varrho) \subset S_2$.

Morse-Conley graph



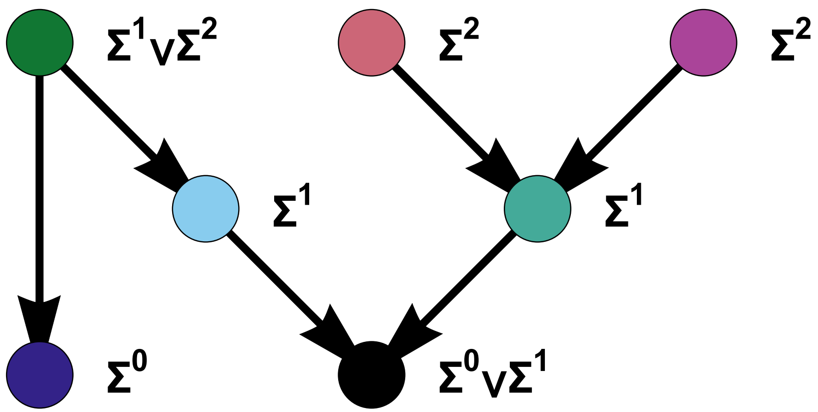


Morse inequalities

Theorem. Given a Morse decomposition $M = \{ M_\iota \mid \iota \in P \}$ of an isolated invariant set S we have

$$\sum_{\iota \in P} p_{M_\iota}(t) = p_S(t) + (1+t)q(t)$$

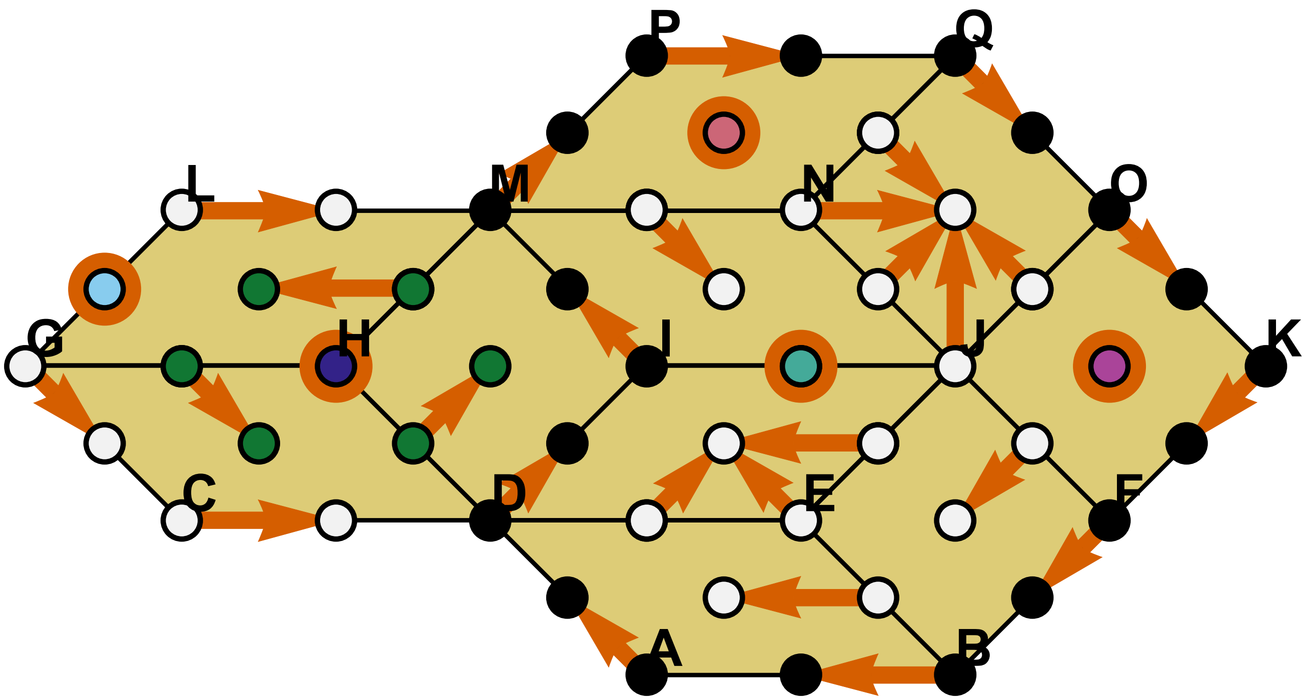
for some non-negative polynomial q.



$$\begin{aligned} p_1(t) &= 1 \\ p_2(t) &= 1+t \\ p_3(t) &= t \\ p_4(t) &= t \\ p_5(t) &= t+t^2 \\ p_6(t) &= t^2 \\ p_7(t) &= t^2 \end{aligned}$$

$$\sum_{\iota \in P} p_{M_\iota}(t) = 2 + 4t + 3t^2 = 1 + (1+t)(1+3t) = p_S(t) + (1+t)q(t)$$

Concise Morse decomposition



In the setting of CW complexes: the cells not in a Morse set may be quotiented out.

Relation to classical theory

$\mathcal{X}$ - the collection of cells of a CW complex $X = \bigcup \mathcal{X}$.

Conjecture. Given a Morse decomposition

$$\mathcal{M} = \{ \mathcal{M}_p \mid p \in P \}$$

of $\mathcal{X}$, there exists a flow φ on X and a Morse decomposition $M = \{ M_p \mid p \in P \}$ of φ such that for any interval I in P the Conley indexes of $\mathcal{M}(I)$ and $M(I)$ coincide.

Theorem. (T. Kaczynski, MM, Th. Wanner)
Assume $\mathcal{X}$ is the collection of cells of a simplicial complex $X = \bigcup \mathcal{X}$. Given a Morse decomposition $\mathcal{M} = \{ \mathcal{M}_p \mid p \in P \}$ of $\mathcal{X}$, there exists an usc, acyclic valued, homotopic to identity, multivalued map $F : X \rightrightarrows X$ and a Morse decomposition $M = \{ M_p \mid p \in P \}$ of the induced multivalued dynamical system such that for any interval I in P the Conley indexes of $\mathcal{M}(I)$ and $M(I)$ coincide.

Conclusion and future work

- Morse-Conley theory for combinatorial multivector fields may be constructed.
- It resembles in many, but not all aspects the classical theory.
- It provides a very concise description of dynamics.

- isolating neighborhoods?
- properties of the Conley index
- connection matrix theory
- concise approximation of classical dynamics
- time-discrete dynamical systems