Discrete Morse functions on infinite complexes

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joint work with
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That is, for each cell $\sigma^{(k)}$ one of the following is true

- $F(\tau) \geq F(\sigma)$ for exactly one face $\tau^{(k-1)} < \sigma$
- $F(\sigma) \geq F(\nu)$ for exactly one coface $\nu^{(k+1)} > \sigma$
- $F(\tau) < F(\sigma) < F(\nu)$ for all faces τ and cofaces ν.

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There is an alternative nice definition due to Benedetti ...

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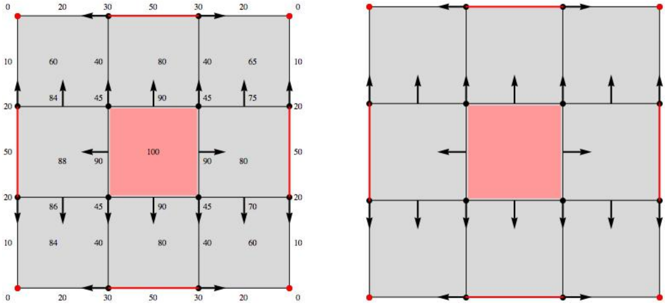
Cells that appear in V are *reguar* and cells that do not are *critical*.

Example 1

Here is an example of a discrete Morse function and the induced gradient vector field on a torus:



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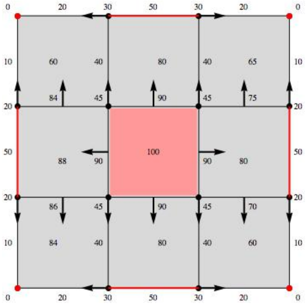
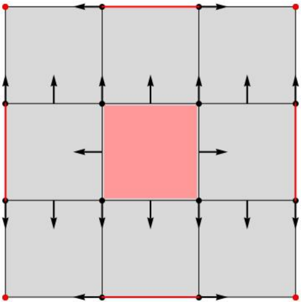
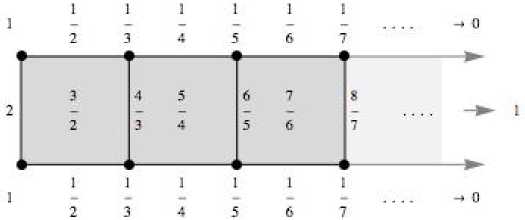




A discrete Morse function on a finite cell complex has at least one critical vertex, at the minimal value.

Example 2

Here is an example on an infninite strip:



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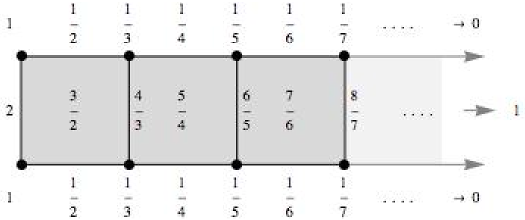

On an infinite complex a discrete Morse function can have no critical cells.

Gradient paths

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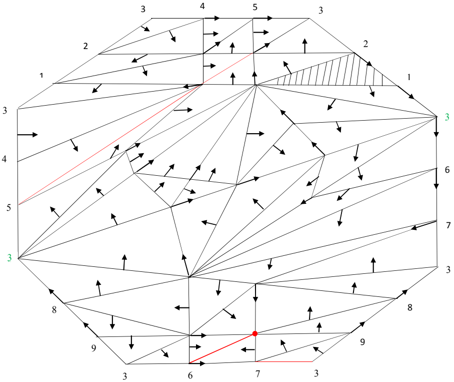

This is a discrete vector field on a genus 2 surface with many *V*-paths.

Discrete vector fields

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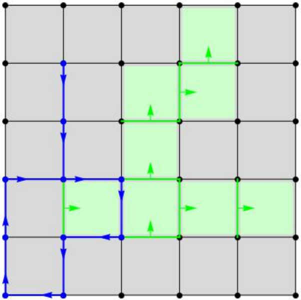

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It works also for infinite locally finite complexes.

Morse inequalities on finite complexes

The following result is well known:

Theorem (Forman)

If F is a discrete Morse function on M with c_k critical cells of dimension k and b_k is the k-th Betti number of M, $k = 0, 1, \ldots, n$ (n is the dimension of M). Then:

1. $c_k \geq b_k$ *for all* k,
2. $c_k - c_{k-1} + \cdots \pm c_0 \geq b_k - b_{k-1} + \cdots \pm b_0$, *for all* k,
3. $c_0 - c_1 + \cdots + (-1)^n c_n = b_0 - b_1 + \cdots + (-1)^n b_n = \chi(M)$.

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Example 2 above shows that Mores inequalities do not hold in general on infinite complexes.

Rays

V a discrete vector field.

A k-ray in V is an infinite sequence

$$\tau_0^{(k-1)} < \sigma_0^{(k)} > \tau_1^{(k-1)} < \sigma_1^{(k)}, \ldots$$

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Two rays are *equivalent* if they coincide from some common cell on.

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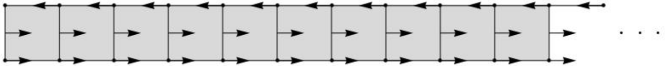

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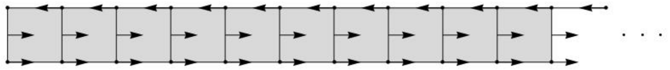

A descending ray is an infinite gradient path.

Critical elements

Theorem (Ayala, Fernándes, Vilches)

If F is a discrete Morse function on an infinite locally finite regular cell complex M with c_k critical cells and d_k equivalence classes of descending rays, and if b_k is the k-th Betti number of M, $k = 0, 1, \ldots, n$ (n is the dimension of M). Then:

1. $(c_k + d_k) - (c_{k-1} + d_{k-1}) + \cdots \pm (c_0 + d_0) \geq b_k - b_{k-1} + \cdots \pm b_0$ *for all* $k = 0, 1, \ldots, n-1$,
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c_i and d_i are all finite in the above theorem.

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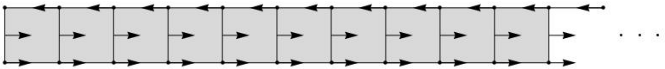

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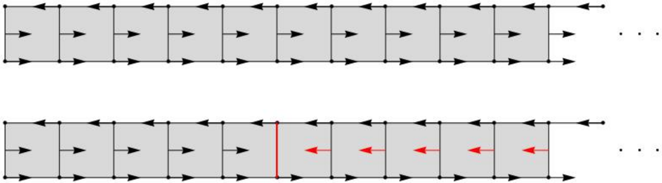

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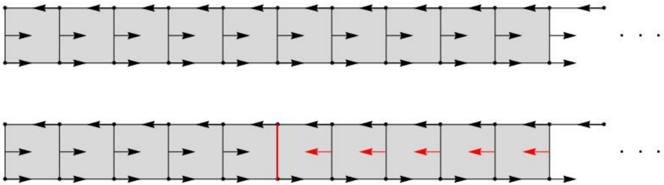

The descending ray is replaced by the critical cell τ and an ascending ray.

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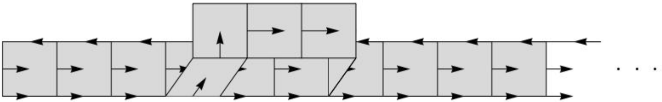

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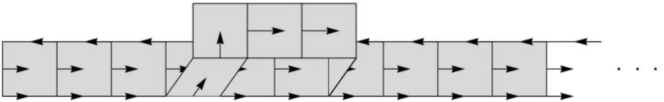

Kukieła (in the more general context of Morse matchings on posets) defined the concept of a *multiray*, which is a ray along which there are infinitely many bypasses, and showed that multirays induces an infinite number of equivalence classes of rays.

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Theorem (Forman's Main theorem)

If M is finite, then

- *if $F^{-1}([a,b])$ contains no critical cells of F then $M(b)$ collapses onto $M(a)$,*
- *if $F^{-1}([a,b])$ contains one critical cell of dimension k then $M(b)$ is homotopy equivalent to $M(a)$ with a k-cell attached along its boundary.*

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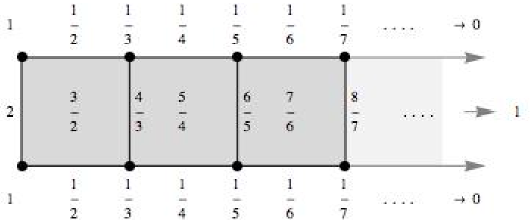
In particular, M has the homotopy type of a CW complex with one cell of dimension k for each critical cell of F of dimension k.

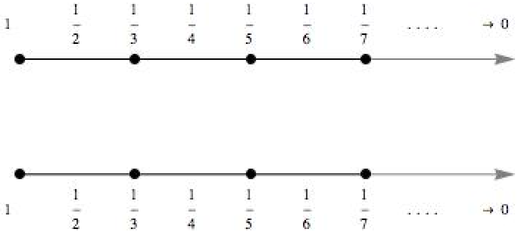
Example

$M = M(2)$



F has no critical cells, in particular no critical cell in $F^{-1}([1,2])$, but $M(1)$ is not homotopy eqivalent to $M(2)$.

$M(1)$



Proper discrete Morse functions

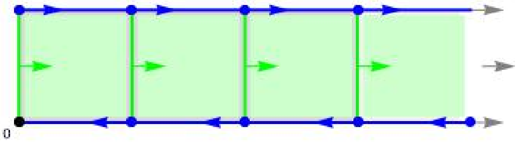
The discrete Morse function F is *proper* if $F^{-1}([a, b])$ contains at most finitely many cells for any interval $[a, b]$. In this case Forman's main theorem easily generalizes:

For a proper discrete Morse function Forman's Main theorem is valid.

Existence of proper integrals

Which discrete vector fields have proper integrals?

No V-loops does not suffice:



For any interval $[-a, a]$, $F^{-1}([-a, a])$ is infinite.

Incident rays

The *descending region of a k-cell* σ consists of all k dimensional V-paths beginning in the boundary of σ. In addition, we add recursively all regular pairs (τ, ν) of lower dimension in the boundary of their union with all cofaces of τ except ν already included.

The *descending region of a ray* is the union of the descending regions of its cells.

A ray r_1 of dimension d_1 is *incident* to a ray r_2 of dimension $d_2 > d_1$ when $D(r_1) \cap \overline{D(r_2)}$ contains infinitely many cells.

Classification theorem

A *forbidden configuration* is a descending ray with an incident ascending ray of lower dimension in the boundary of its descending region.

Theorem (Ayala, Jerše, M, Vilches)

A discrete vector field on a locally finite infinite regular cell complex M with finitely many critical elements admits a proper integral if and only if it has no forbidden configurations.

On complexes of dimension 1 (graphs) no V-loops suffices.

Proof

The proof is an algorithm for constructing such an integral:

1. all critical cells are given the value equal to their dimension,
2. M is expressed as the union $M = \cup_{i=0}^{\infty} K_i$ of an increasing sequence of finite sub complexes with the property, that each K_i intersects any ray in only one component, and that it includes the whole closed descending region of a cell that does not belong to any ray,
3. F is defined on K_0 essentially by carefully assigning decreasing values along V-paths,
4. F is inductively extended from K_i to K_{i+1} in the same way.

Homology

Theorem (Kukieła)

If M is a regular cell complex (not necessarily locally finite) with an acyclic discrete gradient field V that has a finite number of equivalence classes of maximal descending rays then there exists an acyclic gradient field V' with no descending rays, with one critical cell of dimension k for each critical cell of V of the same dimension and in addition a critical cell of dimension k for each descending k-ray of V.

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Actually the theorem is proved in the more general setting of matchings on posets.

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The *Berstein class* b is an element of $H^1(G; I)$, where $I = \ker \mathbb{Z}[G] \to \mathbb{Z}$ is the augmentation ideal.

The grope (by Aleš Vavpetič)

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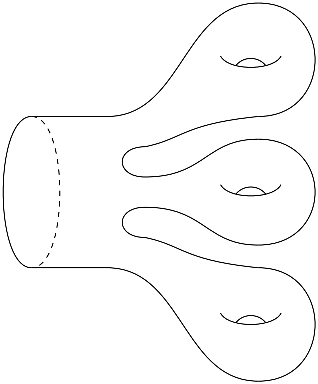

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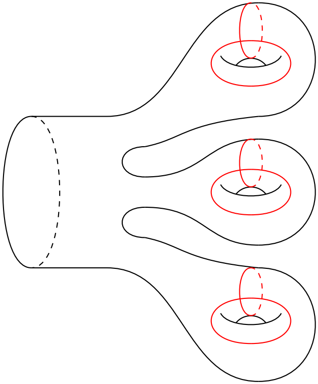

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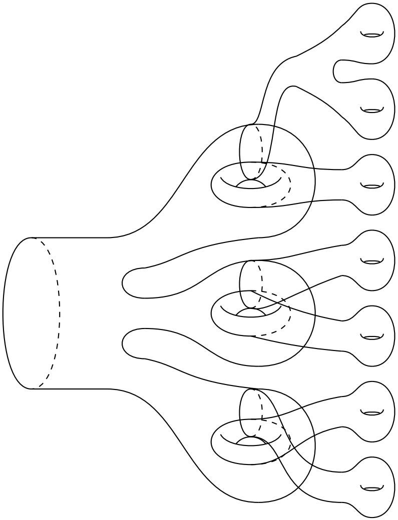

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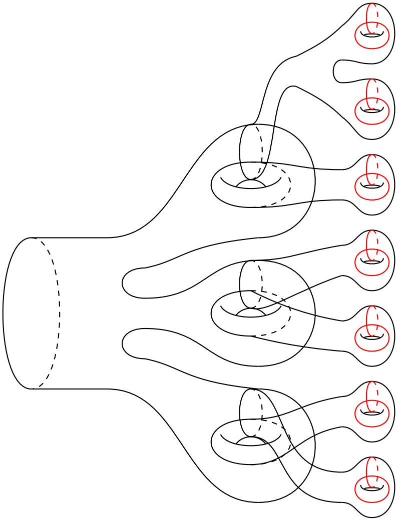

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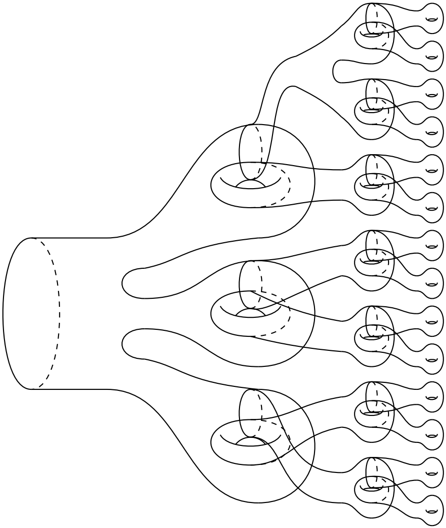

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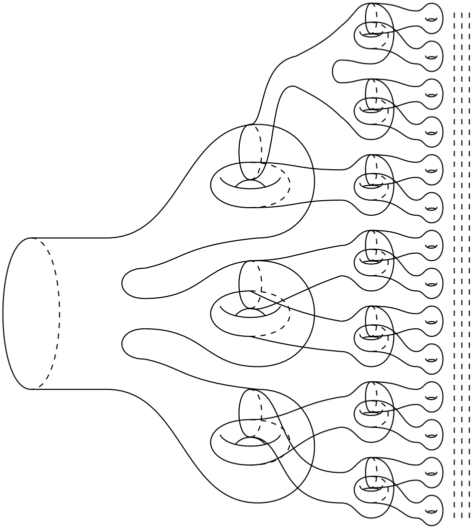

A discrete vector field on the grope

The homology groups H_1 and H_2 are trivial but . . .

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a discrete vector field on the grope has infinitely many descending rays.

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a discrete vector field on the grope has infinitely many descending rays.

Could we obtain a nontrivial dimension two cohomology element from (some version of) discrete Morse theory?

Thank you!