On Computability and Triviality of Well Groups

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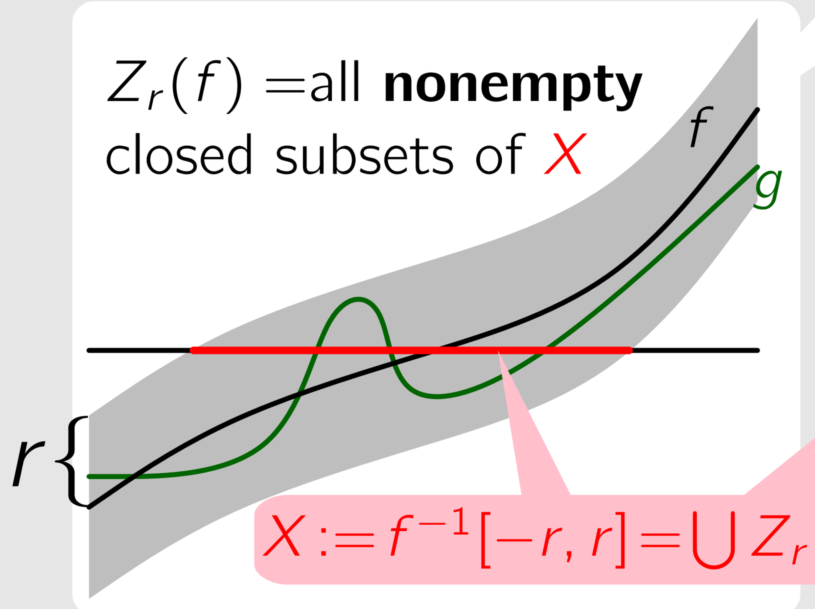
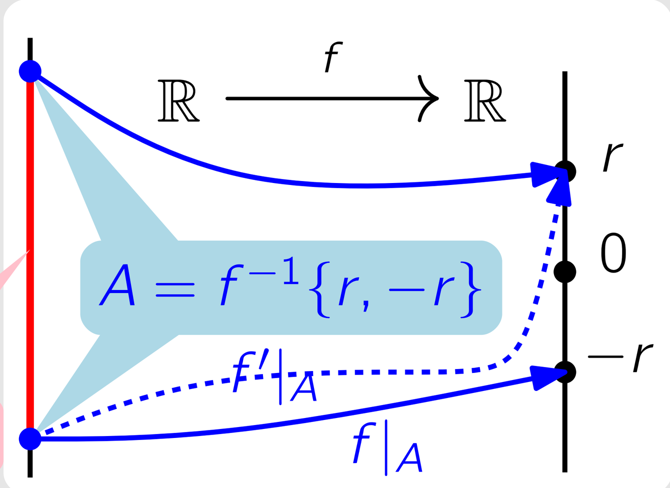
Given $f\colon K \to \mathbb{R}^n$ on a simpl. compl. and a *precision level* $r > 0$

We study: $Z_r(f) := \{g^{-1}(0) \mid g\colon K \to \mathbb{R}^n \text{ s. t. } \underbrace{\|f-g\|}_{\max_x |f(x)-g(x)|} \le r\}$

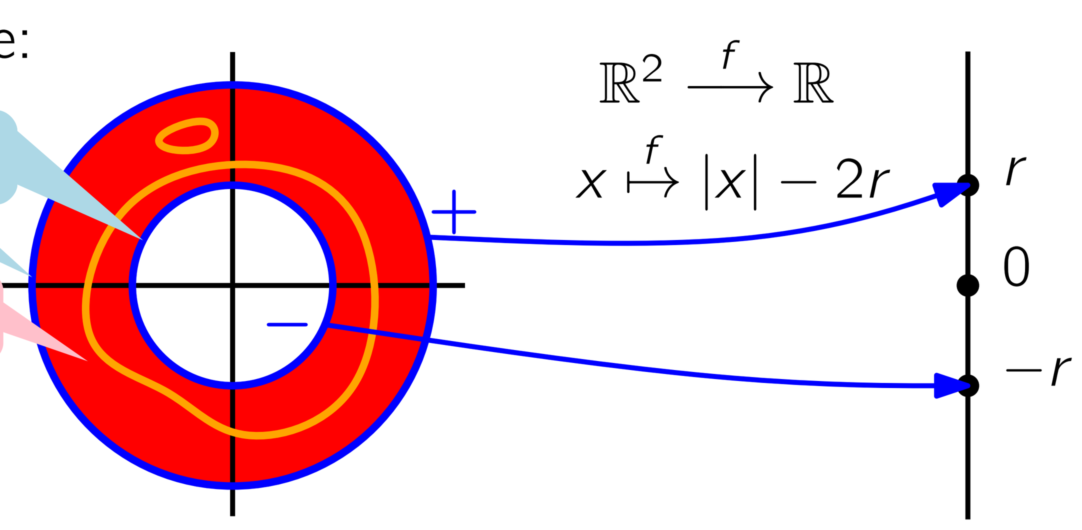
r-perturbation of f



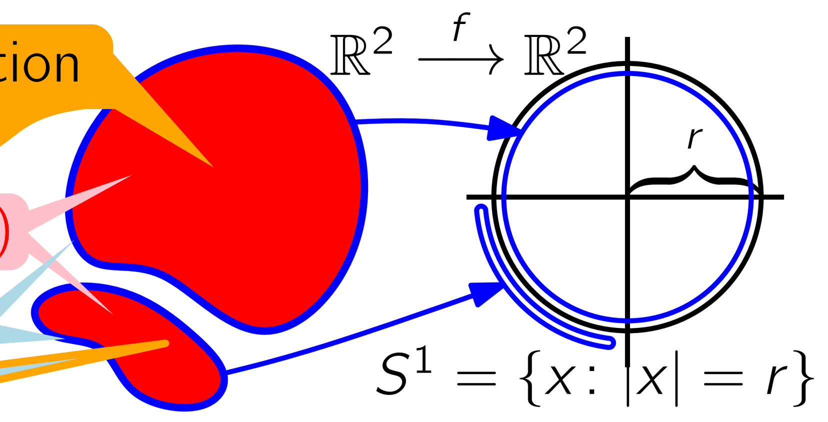
The same example but different pictures:



Another example:



$f|_A$ is crucial:
Thm[FK13]:
$\emptyset \in Z_r(f)$ iff $f|_A$ extends to a map $X \to S^{n-1}$



Outline

We want to study robust properties of zero sets of continuous maps: what can we tell about $g^{-1}(0)$ if only approximation of g is known?

The most established robust descriptor of the zero set so far is *well groups* [EMP11].

Our contribution is twofold:

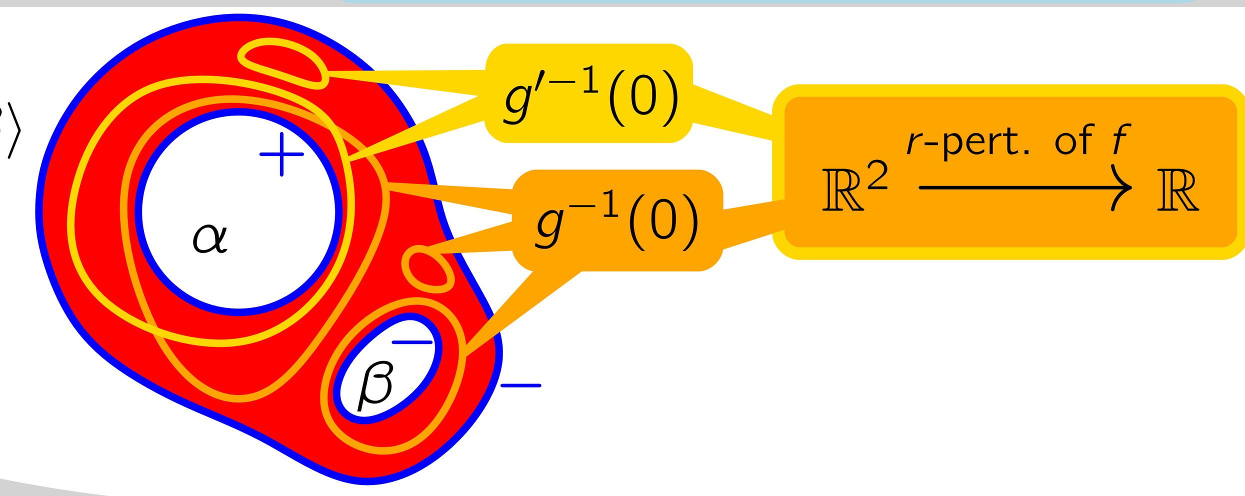
1. We improve on computability of well groups—prove that Patel's computable *new well groups* approximate well groups from below.
2. We show *incompleteness results*—examples where well groups fail to capture important robust features

informally, well groups capture "homological properties" common to zero sets of all r-perturbations of f

well group $U_k(f,r) := \bigcap_{g:\|g-f\|\le r} \{\alpha \in H_k(X) \text{ supported by } g^{-1}(0)\}$

represented by a cycle in $Z_k(g^{-1}(0))$

Example:
$H_1(X) = \langle \alpha, \beta \rangle$
$U_1(f,r) = \langle \alpha \rangle$



down side:
algorithm known only for $n=1$ [BEMP10] and $\dim K = n$ [CPS12]

input encoding: f is simplex-wise linear, $\mathbb{Q}$-valued on vertices

$U_0(f,r)$ computationally equivalent to extendability, key element:

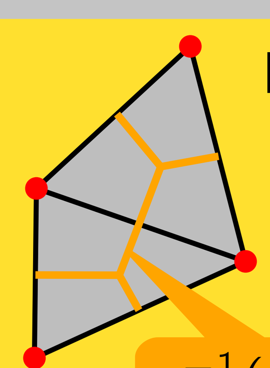
Thm[FK13]: $U_0(f,r) = 0 \Leftrightarrow f|_A$ extends to a map $X \to S^{n-1}$

- algorithm for $\dim K \le 2n-3$ or $n = 1, 2$ or n even
- undecidable otherwise

Extendability is also relevant for higher well groups:

Lemma: If $f|_A$ can be extended to a map $X^{(i-1)} \to S^{n-1}$ then $U_j(f,r) = 0$ for $j > m - i$

$(i-1)$-skeleton of X



Idea of the proof
$X^{(i-1)} \to S^{n-1}$
$\Downarrow$
$g\colon X \to \mathbb{R}^n$

$g^{-1}(0)$ is "dual" to $X^{(i)}$

Is the "lack of extendability to $X^{(i)}$" related to well groups?

...YES when $i = n$ (extension $X^{(n-1)} \to S^{n-1}$ always exists)

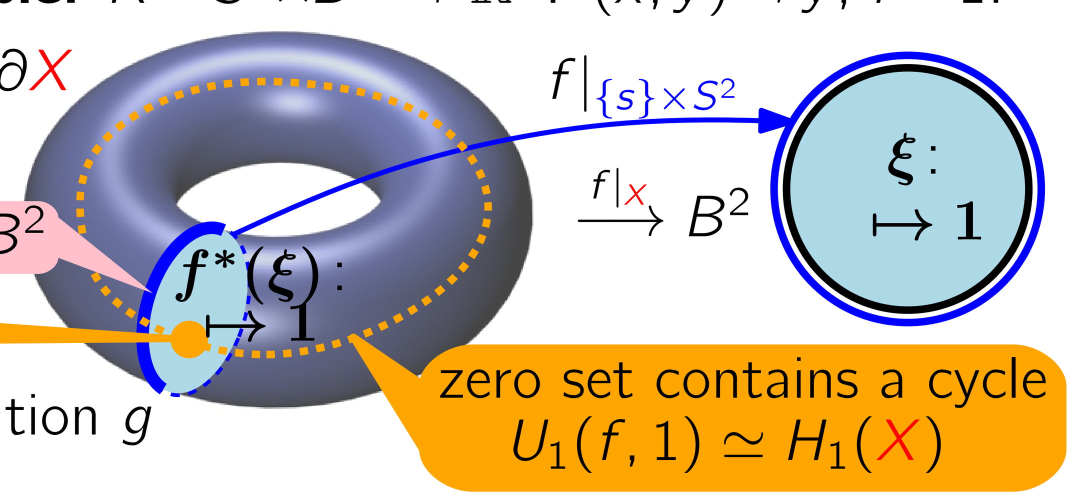
nonexdendability to $X^{(n)}$ measured by "primary obstruction" ϕ
computable: $\phi = f^*(\xi)$ where $f\colon (X,A) \to (B^n, S^{n-1})$

the fundamental class in $H^n(B^n, S^{n-1})$

Thm 1: $\phi \frown H_*(X,A) \subseteq U_{*-n}(f,r)$

Solid torus example: $K = S^1 \times B^2 \xrightarrow{f} \mathbb{R}^2\colon (x,y) \overset{f}{\mapsto} y,\ r = 1$:
then $X = K$, $A = \partial X$

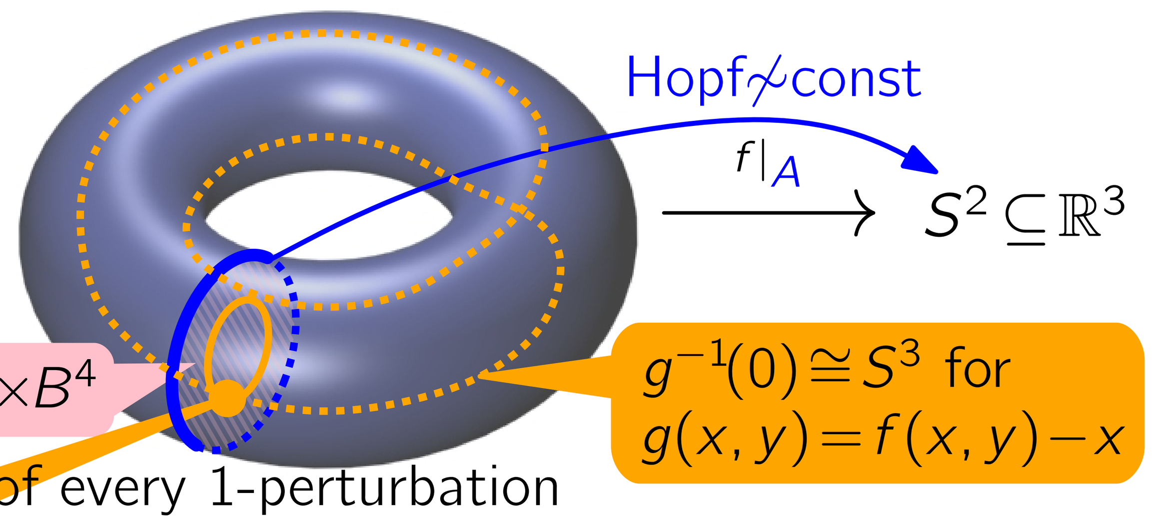
Each section $\{s\} \times B^2$ contains a root of every 1-perturbation g



...NO (in some sense) when $i > n$, i.e., $\phi = 0$, example:

solid hyper-torus: $K = S^2 \times B^4 \xrightarrow{f} \mathbb{R}^3\colon (x,y) \overset{f}{\mapsto} |y|\mathrm{Hopf}(y/|y|)$

$A = S^2 \times S^3$
$X = S^2 \times B^4$



Each section $\{s\} \times B^4$ contains a root of every 1-perturbation

Thm 2: Well groups of f are trivial except for $U_0(f,r)$. Moreover, there is f' with $U_*(f',r) = U_*(f,r)$ but having only single root (robustly), i.e., $Z_r(f) \ne Z_r(f')$

Thm 2': Thm 2 generalizes to $X = S^{m-i} \times B^i$ where $m - i < n$, $n < i < (m+n)/2$ and $\pi_{i-1}(S^{n-1})$ and $\pi_{m-1}(S^{n-1})$ are nontriv.

More general triviality results, but we omit it here

Conjecture: if the primary obstruction ϕ vanish, then $U_j(f,r) = 0$ for $j = m-n-1, \ldots, (m-n+2)/2$

A further direction:

Thm. $Z_r(f)$ is determined by X,A and the homotopy class of $f|_A\colon A \to S^{n-1}$

$\Rightarrow$ complete invariant when zero sets are equipped by framing

References:

[BEMP10] *The Robustness of Level Sets* by Bendich, Edelsbrunner, Morozov and Patel
[CPD12] *Computing the robustness of roots* by Chazal, Patel and Skraba
[EMP11] *Quantifying transversality by Measuring the Robustness of Intersections* by Edelsbrunner, Morozov and Patel
[FK13] *Robust satisfiability of systems of equations* by Franek and Krčál