Higher-dimensional Discrete Cheeger Inequalities

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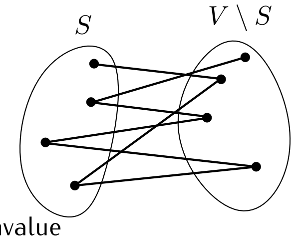
Graphs

Edge expansion:

$$h(G) := \min_{S\subset V} \frac{|V||E(S,V\setminus S)|}{|S||V\setminus S|}$$



Spectral expansion:

$\lambda(G) := \lambda_2(L) =$ second smallest eigenvalue of Laplacian $L = D - A$

Discrete Cheeger Inequality

$$G\ d\text{-regular: } \lambda(G) \le h(G) \le \sqrt{8d\lambda(G)}$$

[Dodziuk, Alon-Milman - 1980s]

Generalization to higher-dimensional complexes?

Spectral Expansion for **2-Complexes**

Higher-dimensional Laplacians (Eckmann, 1940s):

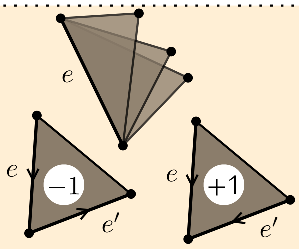
$$L = L_1^{\text{up}} = D - A$$

$$D_{e,e'} = \begin{cases}\deg(e) & e = e'\\ 0 & \text{else.}\end{cases}$$

$\deg(e) = 3$:

$$A_{e,e'} = \begin{cases}\pm 1 & e \sim e'\\ 0 & \text{else.}\end{cases}$$

$e \sim e'$:



Fix orientation of edges!

- Connected to $\mathbb{R}$-cohomology: $L = \partial_2\delta_1$
- $\lambda(X) :=$smallest *non-trivial* eigenvalue of L $=$ smallest eigenvalue on $(B^1(X))^\perp$

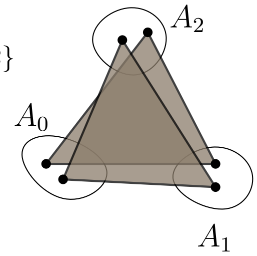
Combinatorial Expansion for **2-Complexes**

A very natural generalization:

$F(A_0, A_1, A_2) := \{t \in T : |t \in A_i| = 1 \text{ for } i = 0,1,2\}$

$A_0 \dot\cup A_1 \dot\cup A_2 = V$

$$h(X) := \min_{A_0\dot\cup A_1\dot\cup A_2 = V} \frac{|V||F(A_0,A_1,A_2)|}{|A_0||A_1||A_2|}$$



(assume X has complete 1-skeleton)

Vertices vs. triangles!

Different generalization: Edges vs. triangles!

Based on $\mathbb{Z}_2$-cohomology:

$$h_{\mathbb{Z}_2}(X) := \min_{A\subseteq E} \frac{|V||\delta_X A|}{|\delta_{K_n^2} A|}$$

$\delta_X A = \{t \in T \colon e$ has an odd number of edges from $A\}$

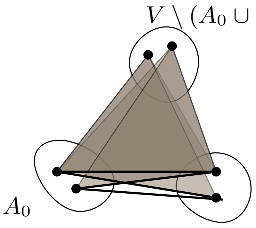
Emerged in various contexts:

Gromov, Linial&Meshulam, Newman&Rabinovich

Comparing...

$$h(X) = \min_{\substack{A_0,A_1\subset V,\\ A_0\cap A_1 = \emptyset}} \frac{|V||\delta_X E(A_0,A_1)|}{|\delta_{K_n^2} E(A_0,A_1)|}$$

$$\Rightarrow h_{\mathbb{Z}_2}(X) \le h(X)$$



Theorem 1: X with complete 1-skeleton:

$$\lambda(X) \le h(X)$$

[Parzanchevski, Rosenthal, Tessler - 2012]

$\lambda(X) \le h_{\mathbb{Z}_2}(X)$? Not possible!

[G., Wagner - 2012]
[Steenbergen, Klivans, Mukherjee - 2012]

Extension of Theorem 1

Basic idea of [Parzanchevski, Rosenthal, Tessler - 2012] allows extension:

$$h'(X) := \min_{\substack{A\subset E(A_0,A_1),\\ A_0,A_1\subset V}} \frac{|V|\cdot|\delta_X A|}{|\delta_{K(X)} A|}$$

Theorem 2: X any 2-complex:

$$\lambda(X) \le h'(X)$$

[G., Szedlák - 2014]

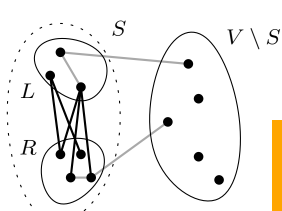
- $h'(X)$ like $h_{\mathbb{Z}_2}(X)$ but only for subsets of cuts.
- As $h'(X) \le h(X)$: implies Theorem 1 for non-complete 1-skeleton

The other end of the spectrum

For **graphs**, largest eigenvalue of Laplacian related to bipartiteness:

$$\beta(G) := \max_{S\subset V, L\sqcup R = S} \frac{2|E(L,R)|}{\sum_{v\in S}\deg(v)}$$

"bipartiteness ratio"



$$2 - \lambda_n(\Delta) \le 2(1 - \beta(G)) \le 2\sqrt{2(2 - \lambda_n(\Delta))}$$

[Trevisan - 2012]

For **complexes**:

$$\beta(X) := \max_{\substack{S\subset V,\\ A_0\sqcup A_1\sqcup A_2 = S}} \frac{3|T(A_0,A_1,A_2)|}{\sum_{e\in E(S)}\deg(e)}$$

$$3 - \lambda_{\max}(\Delta_1^{\text{up}}) \le 3(1 - \beta(X))$$

[G. - 2013]

Open questions

- Upper bound for $h(X)/h'(X)/h_{\mathbb{Z}_2}(X)$ in terms of $\lambda(X)$?
- Computable lower bound for $h_{\mathbb{Z}_2}(X)$?
- Upper bound at other end of spectrum?