THE UNIMODAL p-CATEGORY IS NOT MONOTONIC IN p

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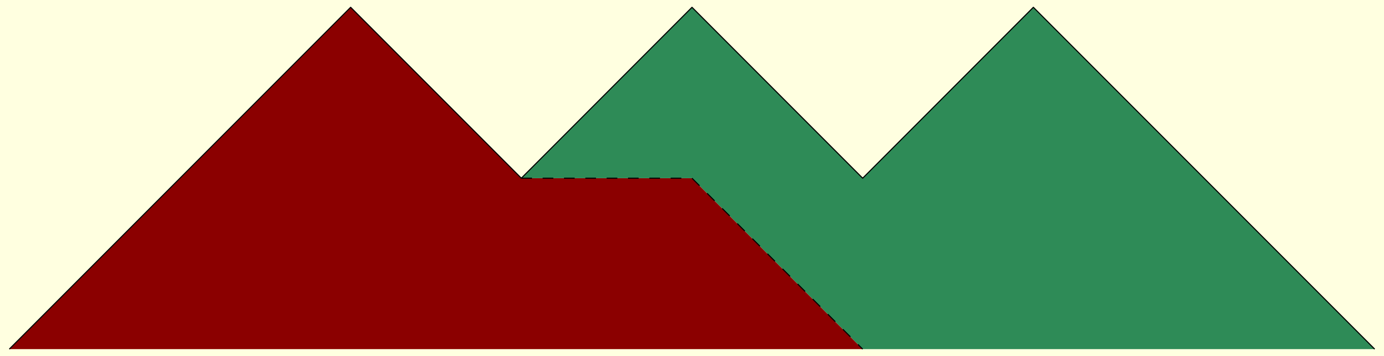
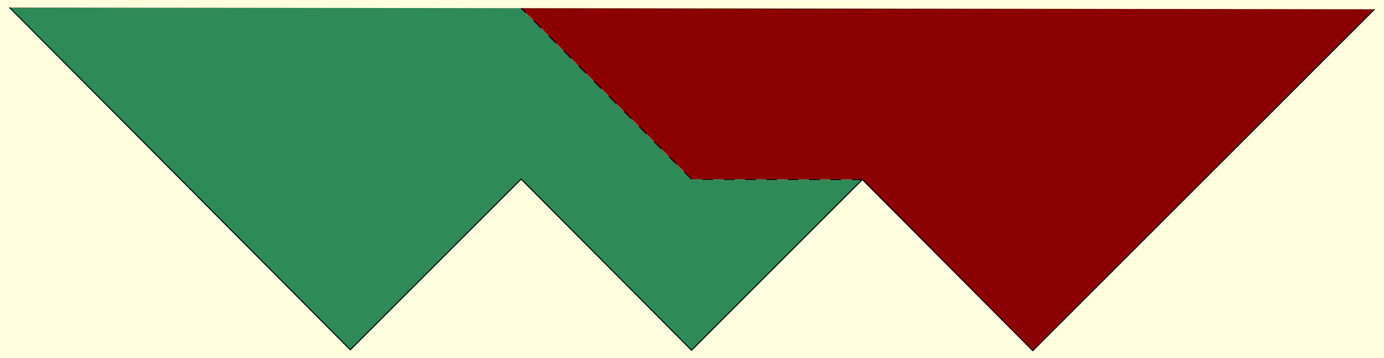
THE QUESTION

Is $\mathbf{ucat}^p(f)$ monotonic in p for a fixed function $f : X \to [0, \infty)$? [1]



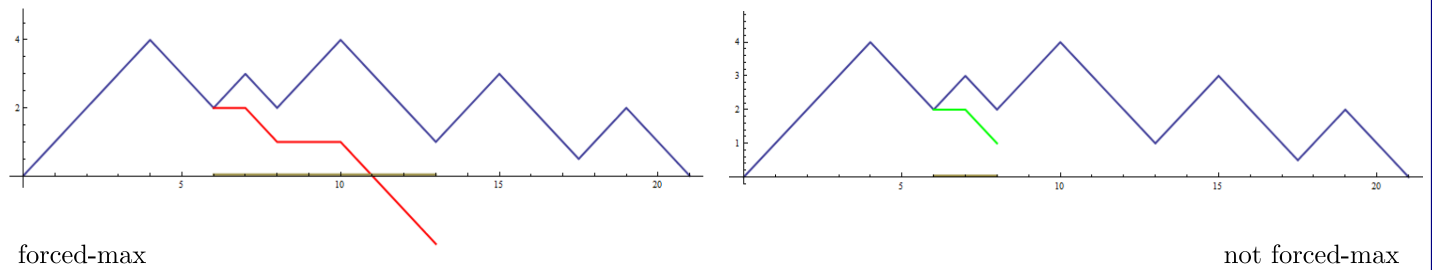


YES: if $X = \mathbb{R}$ or $X = S^1$

Generalized forced-max condition

$[a, b]$ is **forced-max** w.r.t. $f \overset{\text{def}}{\iff} V^-(f; [a, b]) > f(a)$.

(Generalizes the definition of Baryshnikov and Ghrist and makes sense for arbitrary intervals and functions.)



Generalized sweeping

The **sweeping algorithm** of Baryshnikov and Ghrist generalizes to arbitrary functions.

The decomposition is defined by induction:
First define a sequence $(x_i)_i$:

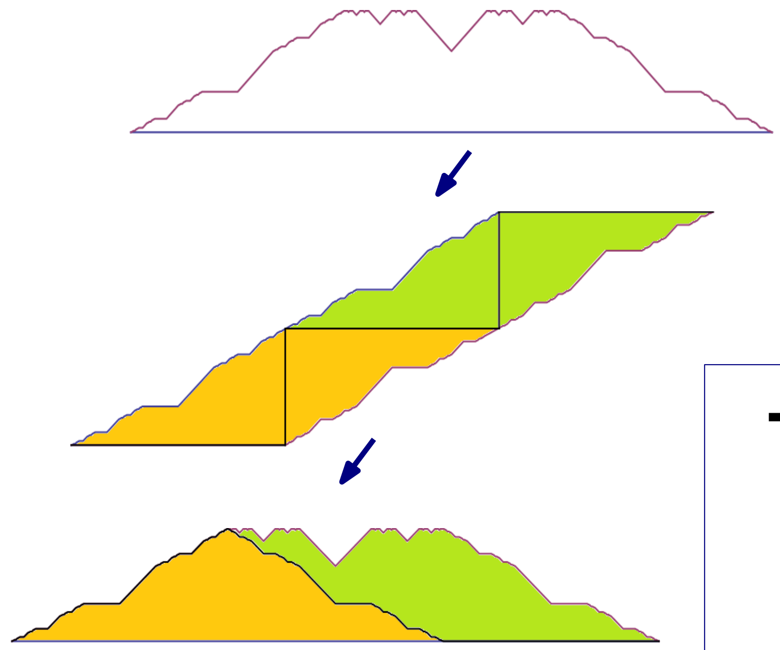
$$x_0 = a,$$
$$x_{i+1} = \inf\{x \mid (x_i, x) \text{ forced-max w.r.t. } f\}.$$

This allows us to define $u_i : \mathbb{R} \to [0, \infty)$ by

$$u_i(x) = \begin{cases} 0; & x \le x_{i-1}, \\ g(x) - g(x_{i-1}); & x \in [x_{i-1}, x_i], \\ h(x_{i+1}) - h(x); & x \in [x_i, x_{i+1}], \\ 0; & x \ge x_{i+1}, \end{cases}$$

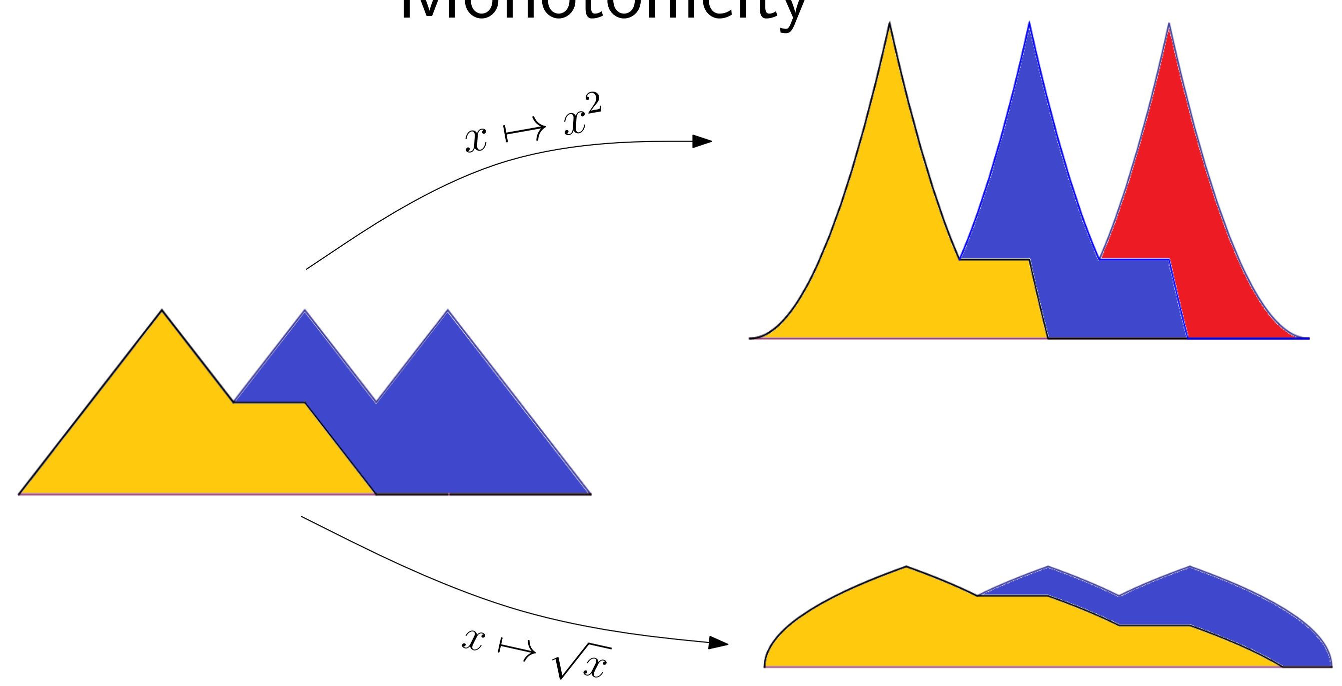
where

$$g(x) = V^+(f; (-\infty, x]),$$
$$h(x) = V^-(f; (-\infty, x]).$$



Observe that $f = g - h$ is precisely the **Jordan decomposition** of f.

Monotonicity

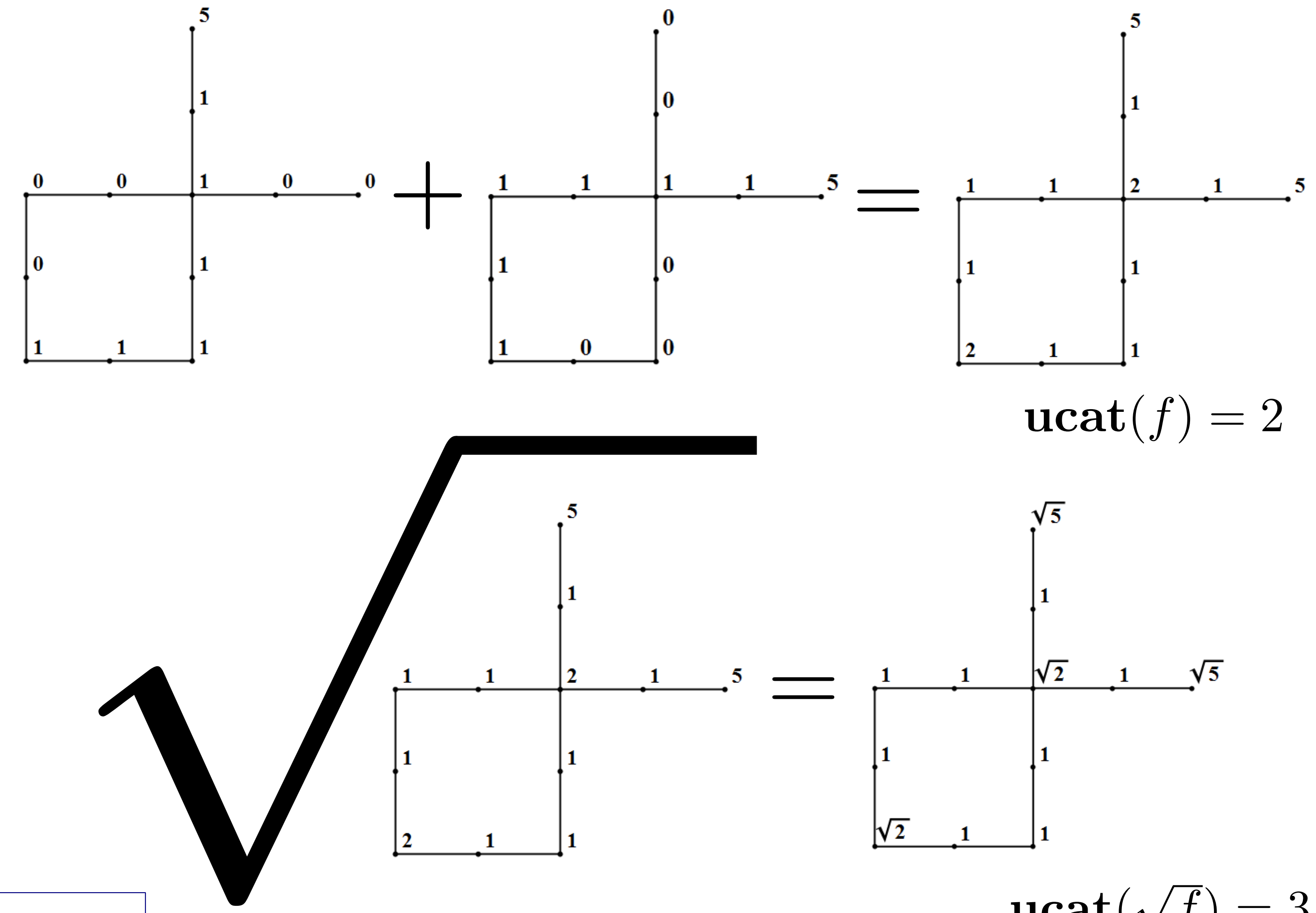


Proof Idea

Suppose $p < q$, it suffices to show that $V^-(f^p; [a, b]) > f^p(a)$ implies $V^-(f^q; [a, b]) > f^q(a)$. For finite sequences (PL functions) this follows from **Karamata's majorization inequality**. In general, the negative variation is a supremum over such sequences. For S^1 the same proof works under a suitable generalization of sweeping.

NO: for more general X

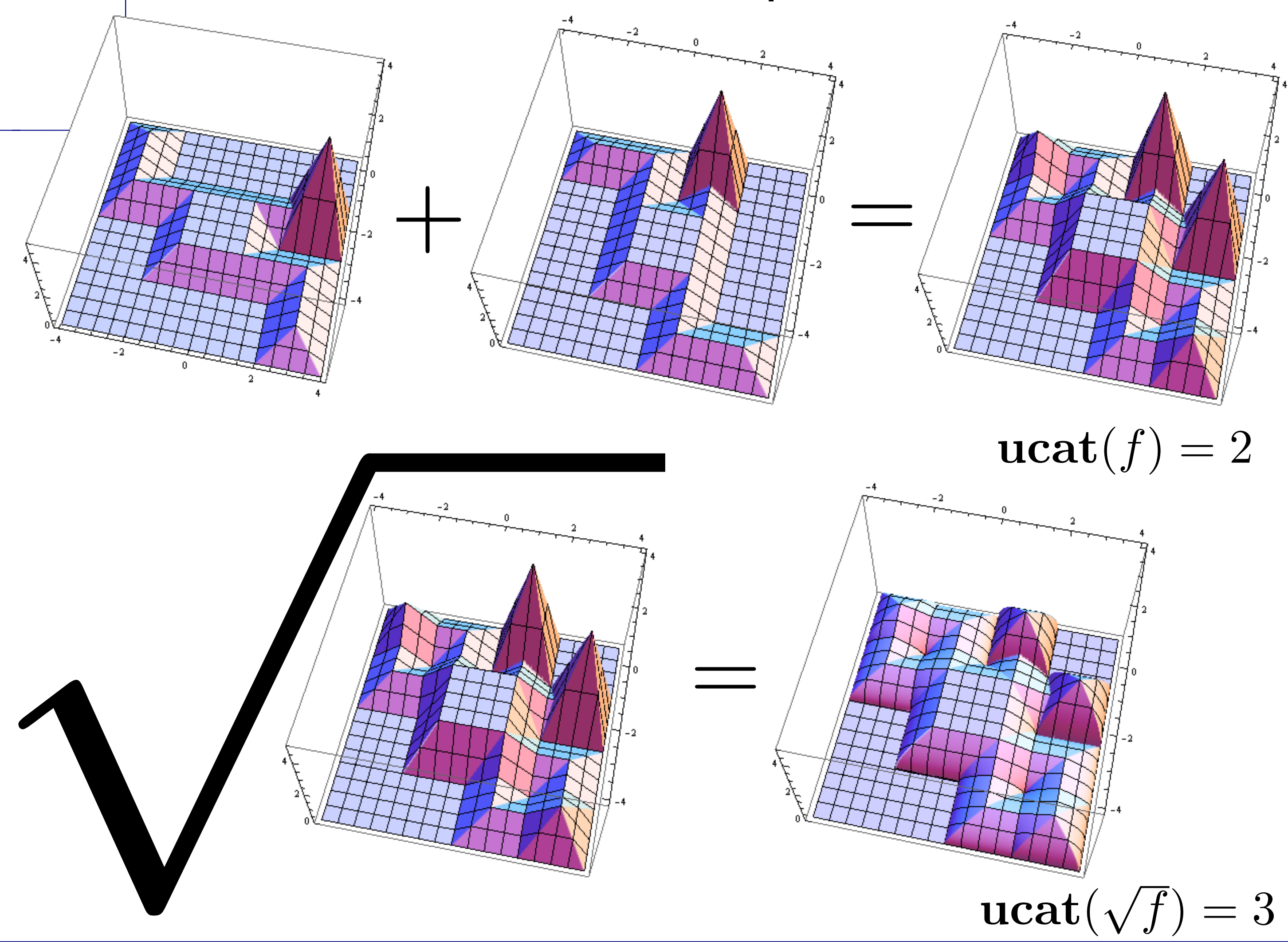
A counterexample on a graph



$\mathbf{ucat}(f) = 2$

$\mathbf{ucat}(\sqrt{f}) = 3$

A counterexample on $\mathbb{R}^2$

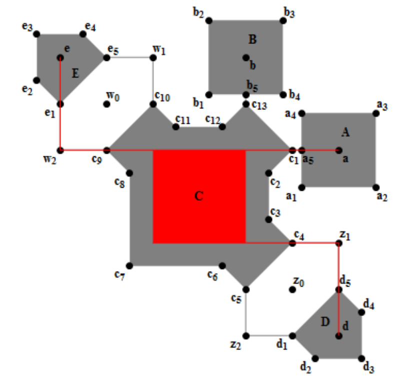


$\mathbf{ucat}(f) = 2$

$\mathbf{ucat}(\sqrt{f}) = 3$

Proof Idea

- suppose $\sqrt{f} = u_1 + u_2$
- u_1 and u_2 achieve their maxima near the two large peaks of f
- $f^{-1}[1, \infty)$ contains **cycles** (picture on the right)
- this forces $u_1(x) = u_2(x) = 1$ for some x away from the peaks
- this contradicts the fact that $\sqrt{f(x)} \le \sqrt{2}$



Definitions [1]

A function $u : X \to [0, \infty)$ is **unimodal** if $u^{-1}[c, \infty)$ is contractible for all $c \in (0, M]$ and empty for $c > M$ (for some $M > 0$).

A **unimodal p-decomposition** of $f : X \to [0, \infty)$ is a decomposition $f = (\sum_{i=1}^n u_i^p)^{\frac{1}{p}}$ into an ℓ^p-combination of unimodal functions. Note that for $p = 1$, we get a sum of unimodal functions. For $p = \infty$, we may use $f = \max_{1 \le i \le n} u_i$.

The **unimodal p-category** $\mathbf{ucat}^p(f)$ is the minimal number of summands in a unimodal p-decomposition of f. Observe that $\mathbf{ucat}^p(f) = \mathbf{ucat}(f^p)$ for $p < \infty$. (We usually write $\mathbf{ucat}(f)$ in place of $\mathbf{ucat}^1(f)$.)

References

[1] Y. Baryshnikov, R. Ghrist. Unimodal Category and Topological Statistics, *Proceedings of NOLTA 2011*.

[2] L. Hickok, J. Villatoro, X. Wang. Unimodal Category of 2-Dimensional Distributions, unpublished.

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TOPOSYS

SEVENTH FRAMEWORK PROGRAMME

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