Are two given maps homotopic? An algorithmic viewpoint

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Results

Abstract

We describe an algorithm that decides the existence of a homotopy between two given maps into a simply connected space. It is based on a computation of the non-commutative group $[\Sigma X, Y]$ of pointed homotopy classes of *continuous* maps from a suspension. Details are found in [6].

Algorithm A

INPUT:
- finite pointed simplicial sets X, Y
- pointed simplicial maps $f, g\colon X \to Y$

OUTPUT:
- decides whether f and g are homotopic

Algorithm B

INPUT:
- finite pointed simplicial sets X, Y

OUTPUT:
- group $[\Sigma X, Y]$ of pointed homotopy classes of maps $\Sigma X \to Y$

Both algorithms require Y to be simply connected. When the dimension of X is fixed, they run in polynomial time.

Previous results

- Analogous results for ***stable homotopy*** were obtained in the series [2, 3, 5] and relied on the computation of the abelian group $\{X, Y\}$ of stable homotopy classes. Moreover, [4] shows that such a computation cannot be made non-stably, at least not for homotopy relative to $A \subseteq X$.
- Here we utilize the computation of the *non-commutative* group $[\Sigma X, Y]$. For X and Y finite, it is a polycyclic group and computations with such groups are possible, see the section about Polycyclic groups.

The following generalizations are proved in [6]:
- relative to a simplicial subset $A \subseteq X$,
- fiberwise over a simplicial set B,
- equivariant with respect to *free* actions of a finite group.

Reductions

- The full strength of Algorithm A requires a fibrewise version of Algorithm B. For simplicity, we therefore restrict to the case that g is the null map, i.e. we describe an algorithm that decides whether f is *nullhomotopic*.
- Since $[X, Y] \cong [X, P_n]$ for $n = \dim X$, we may reduce the algorithms to the case $Y = P_n$. These instances will be called **Algorithm $\mathbf{A}_n$** and **Algorithm $\mathbf{B}_n$**.
- For $f\colon X \to Y$, we denote by f_n the composition $f_n\colon X \xrightarrow{f} Y \xrightarrow{\varphi_n} P_n$.

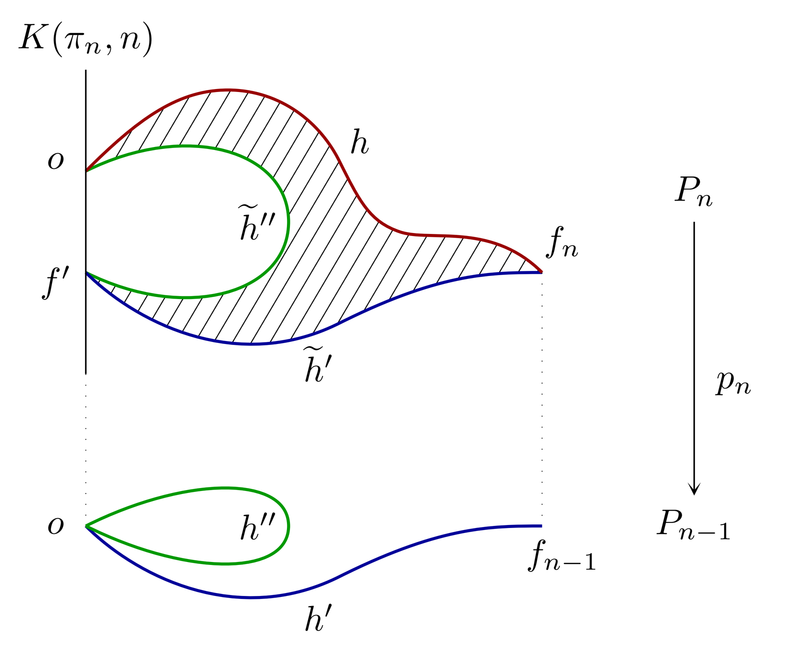
Algorithm A_n (from A_{n-1} and B_{n-1})

The induction is based on the following long exact sequence

$$[\Sigma X, P_{n-1}] \xrightarrow{\partial} [X, K(\pi_n, n)] \xrightarrow{j_*} [X, P_n] \xrightarrow{p_{n*}} [X, P_{n-1}],$$

found as $(*)$ in the section about the Postnikov tower.

- Using Algorithm A_{n-1}, we compute a nullhomotopy $h'\colon o \sim f_{n-1}$, where o denotes the null map.
- Lift h' to some $\widetilde{h}'\colon f' \sim f_n$ using Proposition 1 in the sections about the Postnikov towers. Since $p_n f' = o$, we may interpret f' as a map to the fibre of p_n, i.e. $f'\colon X \to K(\pi_n, n)$.
- We compute generators of $[\Sigma X, P_{n-1}]$ using Algorithm B_{n-1}, decide whether $[f'] \in \operatorname{im} \partial$ and find h'' with $\partial[h''] = [f']$.
- Proposition 1 computes a lift $\widetilde{h}''\colon o \sim f'$.
- The nullhomotopy of f is then given as a concatenation $h = \widetilde{h}' \star \widetilde{h}''$, computed by Proposition 1.



- If either h' or h'' fails to exist, f_n is not nullhomotopic.

Algorithm B_n (from A_{n-1} and B_{n-1})

- Specializing $(*)$ to ΣX, we get a long exact sequence

$$[\Sigma^2 X, P_{n-1}] \xrightarrow{\partial} H^n(\Sigma X, \pi_n) \xrightarrow{j_*} [\Sigma X, P_n] \xrightarrow{p_{n*}} [\Sigma X, P_{n-1}] \xrightarrow{k_{n*}} H^{n+1}(\Sigma X, \pi_n).$$

- Using Algorithm B_{n-1} and the effective computability of cohomology groups, all terms are fully effective polycyclic with the exception of the middle one.
- The algorithm follows from the 5-lemma for polycyclic groups (see the section about Polycyclic groups), once we demonstrate the effective exactness.

second term: given $\alpha \in H^n(\Sigma X, \pi_n)$ in the image of ∂, compute its preimage; this is easily obtained from a list of generators of $[\Sigma^2 X, P_{n-1}]$, given by Algorithm B_{n-1} for $X := \Sigma X$.

third term: given $f_n\colon \Sigma X \to P_n$ with f_{n-1} nullhomotopic, find $f' \sim f_n$ with $\operatorname{im} f'$ lying in the fibre of p_n; first compute a nullhomotopy $h\colon o \sim f_{n-1}$ via Algorithm A_{n-1} and then lift it to a homotopy $f' \sim f_n$ using Proposition 1.

fourth term: given $f_{n-1}\colon \Sigma X \to P_{n-1}$, compute a lift to P_n; achieved by Proposition 1.

Tools

Polycyclic groups

A group G is called ***polycyclic***, if there exists a sequence of subgroups

$$G = G_r \geq G_{r-1} \geq \cdots \geq G_1 \geq G_0 = 0$$

such that:
- each G_{i-1} is a normal subgroup of G_i,
- each $G_i/G_{i-1} \cong \mathbb{Z}/q_i$ is a cyclic group, possibly of infinite order.

Choosing $g_i \in G_i$ that generates G_i/G_{i-1}, and writing G additively, we obtain a bijection

$$\mathbb{Z}/q_1 \times \cdots \times \mathbb{Z}/q_r \longrightarrow G, \quad (z_1, \ldots, z_r) \longmapsto z_1 g_1 + \cdots + z_r g_r.$$

A polycyclic group G is said to be *fully effective polycyclic* if
- its elements have a specified encoding in a computer (generally non-unique!),
- there are algorithms provided that compute the group operations, the bijection $\mathbb{Z}/q_1 \times \cdots \times \mathbb{Z}/q_r \cong G$ and its inverse.

This allows effective computations in G, e.g. solving the word problem.

5-lemma for fully effective polycyclic groups

A sequence $K \xrightarrow{\alpha} G \xrightarrow{\beta} H$ is *effectively exact* if there is an algorithm that computes, for each $g \in \ker \beta$, some $k \in K$ with $g = \alpha(k)$.

Given an effectively exact sequence

$$K' \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow H'$$

with K', K, H, H' fully effective polycyclic, G is also fully effective polycyclic.

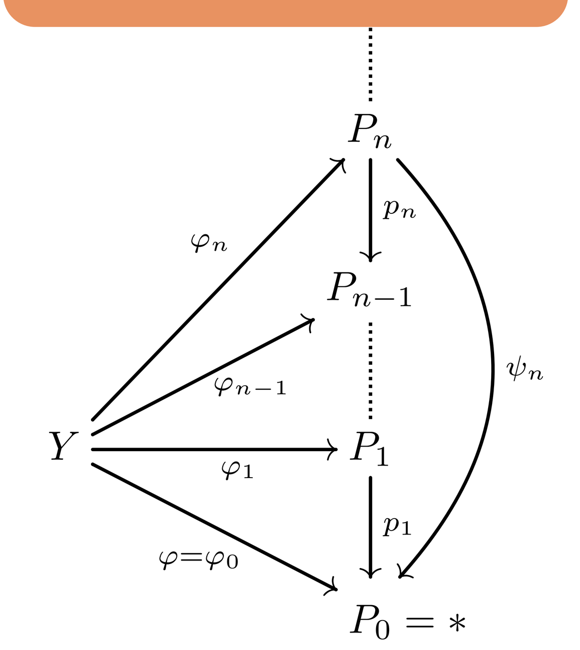
Postnikov tower

Let Y be a simplicial set. A (simplicial) *Postnikov tower* for Y is a commutative diagram satisfying the following conditions:
- The induced map $\varphi_{n*}\colon \pi_i(Y) \to \pi_i(P_n)$ is an isomorphism for $0 \leq i \leq n$ and an epimorphism for $i = n + 1$.
- $\pi_i(P_n) \cong 0$ for $i \geq n + 1$
- There exists a pullback square

$$\begin{array}{ccc} P_n & \longrightarrow & E(\pi_n, n) \\ {\scriptstyle p_n}\downarrow & \lrcorner & \downarrow{\scriptstyle \delta} \\ P_{n-1} & \xrightarrow[k'_n]{} & K(\pi_n, n+1) \end{array}$$

identifying P_n with the pullback $P_{n-1} \times_{K(\pi_n, n+1)} E(\pi_n, n)$. Here $K(\pi_n, n+1)$ is the Eilenberg–MacLane space and $E(\pi_n, n)$ its path space. These have standard simplicial models with $K(\pi_n, n+1)$ a minimal complex and δ a minimal fibration, see [7].

Postnikov stage P_n is an approximation of Y up to dimension n.



Properties

- The paper [3] constructs a Postnikov tower P_n for Y algorithmically; in particular, this algorithm computes the homotopy groups $\pi_n = \pi_n(Y)$.
- The Postnikov stages P_n are generally infinite. On the other hand, they are Kan complexes – maps $X \to P_n$ have a simplicial representative; this representative can be used in algorithmic computations!
- The "difference" between P_{n-1} and P_n is cohomological and as such can be computed:

Long exact sequence

The fibre of p_n, being isomorphic to that of δ, is $K(\pi_n, n+1)$. Thus, we have an exact sequence

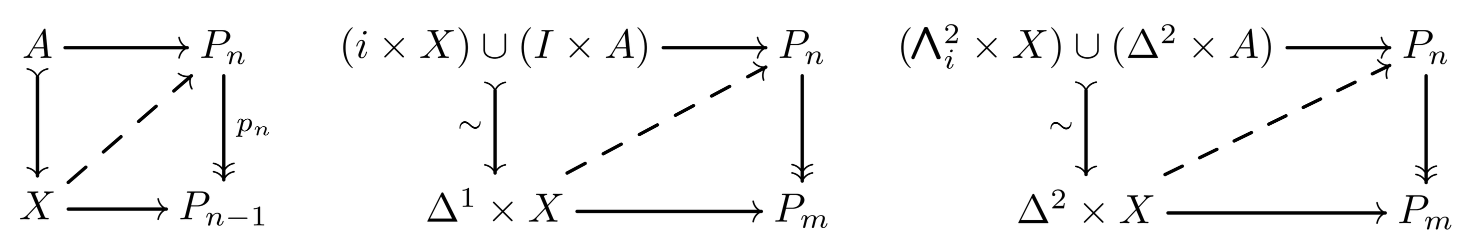
$$[\Sigma X, P_{n-1}] \xrightarrow{\partial} [X, K(\pi_n, n)] \xrightarrow{j_*} [X, P_n] \xrightarrow{p_{n*}} [X, P_{n-1}] \xrightarrow{k_{n*}} [X, K(\pi_n, n+1)]. \qquad (*)$$

In addition, $[X, K(\pi_n, n)] \cong H^n(X; \pi_n)$ and as such is computable; indeed, [2] shows that it is "fully effective abelian" (and thus fully effective polycyclic).

Lifting propositions

Some maps to a Postnikov stage P_{n-1} can be algorithmically lifted to P_n.

Proposition 1 ([5, Propositions 3.5, 3.6, 3.7]). *There is an algorithm that, given one of the diagrams*



where $i \in \{0, 1\}$ or $i \in \{0, 1, 2\}$ respectively, computes a diagonal (if it exists).

The last two cases have a special meaning:
- It is possible to lift homotopies in Postnikov towers algorithmically.
- It is possible to concatenate homotopies in Postnikov towers algorithmically.

References

[1] E. H. Brown (jun.). *Finite computability of Postnikov complexes.* Ann. Math. **65** (1957), 1–20.
[2] M. Čadek, M. Krčál, J. Matoušek, F. Sergeraert, L. Vokřínek, U. Wagner. *Computing all maps into a sphere.* J. ACM **61** (2014), 17:1–44
[3] M. Čadek, M. Krčál, J. Matoušek, L. Vokřínek, U. Wagner. *Polynomial time computation of homotopy groups and Postnikov systems in fixed dimensions.* SIAM J. Comput. **43** (2014), 1728–1780.
[4] M. Čadek, M. Krčál, J. Matoušek, L. Vokřínek, U. Wagner. *Extendability of continuous maps is undecidable.* Discr. Comput. Geom. **51** (2014), 24–66
[5] M. Čadek, M. Krčál, L. Vokřínek. *Algorithmic solvability of the lifting-extension problem.* Preprint, `arXiv:1307.6444`, 2014.
[6] M. Filakovský, L. Vokřínek. *Are two given maps homotopic? An algorithmic viewpoint.* Preprint, `arXiv:1312.2337`, 2013.
[7] J. P. May. *Simplicial Objects in Algebraic Topology.* University of Chicago Press, Chicago, IL, 1992.
[8] P. S. Novikov. *On the algorithmic unsolvability of the word problem in group theory (in Russian).* Trudy Mat. Inst. im. Steklova **44** (1955), 1–143.
[9] C. C. Sims. *Computation with Finitely Presented Groups.* Cambridge University Press, Cambridge, 1994.
[10] J. Rubio, F. Sergeraert. *Constructive homological algebra and applications.* Preprint, `arXiv:1208.3816`, 2012. Written in 2006 for a MAP Summer School at the University of Genova.

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