Advances regarding the persistence of maps

Marc Ethier
Jagiellonian University, Kraków, Poland

The problem

As described by Edelsbrunner, Jabłoński, and Mrozek, we may reconstruct the dynamics of a discretely sampled dynamical system by building a (say Vietoris-Rips) filtration $(K_i)_{i\in\mathbb{Z}}$ on the sample set, constructing partial simplicial maps κ_i extending this dynamical system, and considering the filtered linear maps they induce on the homology spaces $X_i = H_k(\operatorname{dom}\kappa_i)$, $Y_i = H_k(K_i)$ over the field $\mathbb{F}$.
The persistence of (generalized) eigenvectors for an eigenvalue $t \in \mathbb{F}$ along a tower of vector spaces helps us recover important information about dynamics. However, since the filtered maps $\varphi_i = H_k(\kappa_i)$ are not necessarily isomorphisms, we need to also consider, e.g., the map $\psi_i : X_i \to Y_i$ induced by the inclusion of complexes, and compute eigenspaces for pairs of linear maps.

Persistence of eigenvectors

$\mathcal{X} = (X_i, \xi_i)$ and $\mathcal{Y} = (Y_i, \eta_i)$, where ξ_i, η_i are induced by filtration maps, are two towers of finite dimensional vector spaces over $\mathbb{F}$, with morphisms $\varphi, \psi : \mathcal{X} \to \mathcal{Y}$.
For $t \in \mathbb{F}$, we apply the *eigenspace functor* E_t to obtain the tower of eigenspaces:

$$E_t(\varphi_i, \psi_i) = \ker(\varphi_i - t\psi_i).$$

For a tower of finite dimensional vector spaces $\mathcal{U} = (U_i, \upsilon_i)$, we can always build a tower of matchings $\mathcal{B} = (B_i, \beta_i)$ where for every i, B_i is a basis of U_i and β_i is the restriction of υ_i to B_i and B_{i+1}. We can therefore define persistence in the tower of eigenspaces $(E_t(\varphi_i, \psi_i), \epsilon_{t,i})$ where $\epsilon_{t,i}$ is the restriction of ξ_i to $E_t(\varphi_i, \psi_i)$:

$$\begin{array}{ccccccccc}
\dots & \xrightarrow{\epsilon_{t,i-2}} & E_t(\varphi_{i-1},\psi_{i-1}) & \xrightarrow{\epsilon_{t,i-1}} & E_t(\varphi_i,\psi_i) & \xrightarrow{\epsilon_{t,i}} & E_t(\varphi_{i+1},\psi_{i+1}) & \xrightarrow{\epsilon_{t,i+1}} & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \xrightarrow{\xi_{i-2}} & X_{i-1} & \xrightarrow{\xi_{i-1}} & X_i & \xrightarrow{\xi_i} & X_{i+1} & \xrightarrow{\xi_{i+1}} & \dots \\
 & & \downarrow{\scriptstyle \varphi_{i-1}-t\psi_{i-1}} & & \downarrow{\scriptstyle \varphi_i-t\psi_i} & & \downarrow{\scriptstyle \varphi_{i+1}-t\psi_{i+1}} & & \\
\dots & \xrightarrow{\eta_{i-2}} & Y_{i-1} & \xrightarrow{\eta_{i-1}} & Y_i & \xrightarrow{\eta_i} & Y_{i+1} & \xrightarrow{\eta_{i+1}} & \dots
\end{array}$$

A *generalized eigenvector* for the pair (φ, ψ) of maps $X \to Y$ is any vector in a sequence $(x_k)_{k=0}^n \subset X$ where

$$\begin{aligned}
(\varphi - t\psi)(x_0) &= 0,\\
(\varphi - t\psi)(x_1) &= \psi(x_0),\\
&\vdots\\
(\varphi - t\psi)(x_n) &= \psi(x_{n-1}).
\end{aligned}$$

By application of the generalized eigenspace functor we obtain a tower of generalized eigenspaces for the eigenvalue t.
It is known that $E_t(\varphi, \psi)$ can be nontrivial for every value $t \in \mathbb{F}$. To study this, consider $\overline{X}$ the space of polynomial functions from $\mathbb{F}$ to X. If $\alpha : X \to Y$ is a linear map, we can define a linear map

$$\begin{aligned}
\overline{\alpha} : \overline{X} &\to \overline{Y}\\
f &\mapsto \alpha \circ f.
\end{aligned}$$

The operation $(X, \alpha) \to (\overline{X}, \overline{\alpha})$ is a functor in the category of vector spaces over $\mathbb{F}$, and thus we also define the tower of *singular eigenspaces* for the pair of morphisms (φ, ψ):

$$E(\varphi_i, \psi_i) = \ker(\overline{\varphi_i} - t\overline{\psi_i}).$$

The space $E(\varphi_i, \psi_i)$ is a subspace of $\overline{X_i}$, and its image by the evaluation map $e_{t,i} : \overline{X_i} \to X_i$ is a subspace of $E_t(\varphi_i, \psi_i)$. This means that at every i, if $f \in \overline{X_i}$ is a polynomial *singular eigenvector* for the pair (φ_i, ψ_i), then for all $t \in \mathbb{F}$, $e_{t,i}(f) = f(t)$ will be an eigenvector for eigenvalue t.

Kronecker canonical form

Consider A, B matrix representations of maps $\varphi, \psi : X \to Y$, where $\dim X = n$, $\dim Y = m$ over $\mathbb{F}$ algebraically closed. Taking t as a formal variable, a *matrix pencil* is the polynomial matrix $tB - A$. If $m = n$ and $\det(tB - A) \not\equiv 0$, the pencil is called *regular*. By a similarity transformation a regular pencil can be brought to

$$Q_1\,(tB - A)\,R_1 = \operatorname{diag}\{tN - I, tI - J\}$$

where N is a nilpotent Jordan matrix, and J is in Jordan canonical form.
Now suppose that $tB - A$ is *singular* (not regular). If there exists a polynomial solution $f(t) = x_0 + tx_1 + t^2x_2 + \ldots + t^\epsilon x_\epsilon$ to $(tB - A)f(t)$, where $\epsilon \geq 0$ is the minimal degree of such a solution, then $tB - A$ can be brought into

$$Q_2\,(tB - A)\,R_2 = \operatorname{diag}\{L_\epsilon, t\widehat{B} - \widehat{A}\}$$

where $t\widehat{B} - \widehat{A}$ has no polynomial solution of degree less than ϵ and

$$L_\epsilon = \begin{bmatrix} t & -1 & & \\ & \ddots & \ddots & \\ & & t & -1 \end{bmatrix}$$

is a bidiagonal pencil of dimension $\epsilon \times (\epsilon + 1)$.
L_ϵ is called a *column Kronecker block*. Repeatedly splitting from the pencil column Kronecker blocks until all linear dependence on its columns is removed (and then *row Kronecker blocks* L_η^T until all linear dependence on its rows is removed), we end up with a regular pencil, whose canonical form is already known. Therefore, a matrix pencil $tB - A$ can, be a similarity transformation, be brought into the *Kronecker canonical form*

$$Q\,(tB - A)\,R = \operatorname{diag}\{L_{\epsilon_1}, \ldots, L_{\epsilon_p}, L_{\eta_1}^T, \ldots, L_{\eta_q}^T, tN - I, tI - J\}$$

where ϵ_i's are called *column Kronecker indices* and η_i's, *row Kronecker indices*.
For further theoretical details, please refer to Gantmacher.

Algorithm

Following van Dooren, we apply the following algorithm to the pencil $tB - A$.

Algorithm (part 1)
$j := 1$, $A_{1,1} := A$, $B_{1,1} := B$, $m_1 := m$, $n_1 := n$
do
  (* Column reduce the $m_j \times n_j$ matrix $B_{j,j}$ with matrix R_j *)
  $s_j :=$ column nullity (R_j)
  if $s_j = 0$ **do** $l := j - 1$, **return endif**
  $\left[\, B_{j+1} \,|\, 0 \,\right] := B_{j,j}\,R_j$, $\left[\, A_{j+1} \,|\, A_j \,\right] := A_{j,j}\,R_j$
  (* Update other blocks in column j *)
  for $i = 1$ **to** $j - 1$ **do**
    $\left[\, B_{j+1,i} \,|\, B_{j,i} \,\right] := B_{j,i}\,R_j$, $\left[\, A_{j+1,i} \,|\, A_{j,i} \,\right] := A_{j,i}\,R_j$
  endfor
  (* Row reduce and permute rows of the $m_j \times s_j$ matrix A_j with matrix Q_j *)
  $r_j :=$ row rank (Q_j)
  $\begin{bmatrix} 0 \\ \hline A_{j,j} \end{bmatrix} := Q_j\,A_j$, $\begin{bmatrix} A_{j+1,j+1} \\ \hline A_{j+1,j} \end{bmatrix} := Q_j\,A_{j+1}$, $\begin{bmatrix} B_{j+1,j+1} \\ \hline B_{j+1,j} \end{bmatrix} := Q_j\,B_{j+1}$
  $m_{j+1} := m_j - r_j$, $n_{j+1} := n_j - s_j$, $j := j + 1$
repeat

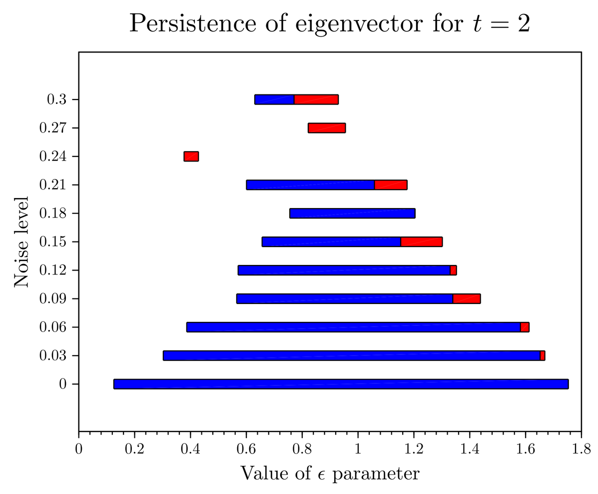
At this point the pencil has been put into block lower triangular form,

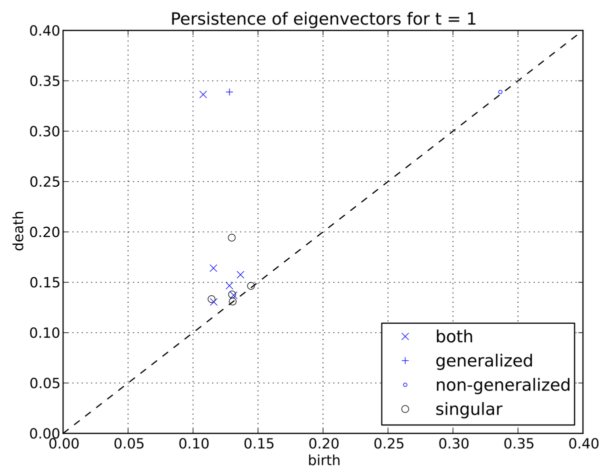
$$\begin{array}{c}
\left[\begin{array}{c|c|c|c|c}
tB_{l+1,l+1} - A_{l+1,l+1} & 0 & \cdots & 0 & 0 \\ \hline
tB_{l+1,l} - A_{l+1,l} & -A_{l,l} & \cdots & 0 & 0 \\ \hline
\vdots & \ddots & \ddots & \vdots & \vdots \\ \hline
tB_{l+1,2} - A_{l+1,2} & tB_{l,2} - A_{l,2} & \cdots & -A_{2,2} & 0 \\ \hline
tB_{l+1,1} - A_{l+1,1} & tB_{l,1} - A_{l,1} & \cdots & tB_{2,1} - A_{2,1} & -A_{1,1}
\end{array}\right]
\begin{array}{l} m_{l+1} \\ r_l \\ \vdots \\ r_2 \\ r_1 \end{array} \\
\begin{array}{ccccc} n_{l+1} & s_l & \cdots & s_2 & s_1 \end{array}
\end{array},$$

where $B_{l+1,l+1}$ has full column rank, the $A_{i,i}$'s have full row rank r_i for $i = 1, \ldots, l$, and the $B_{i,i-1}$'s have full column rank s_i for $i = 2, \ldots, l$. The indices r_i, s_i can be used to obtain the column Kronecker indices: for $i = 1, \ldots, l$, set $s_i - r_i = e_i \geq 0$, and the pencil has e_i column Kronecker blocks of index $i - 1$. Now the blocks $B_{l+1,l+1}$ and $A_{i,i}$, $i = 1, \ldots, l$, can be used to respectively zero out $B_{l+1,i}$ and the A blocks left of $A_{i,i}$, in this order, for i going down from l to 1. This done, the blocks $A_{i,j}$ are used to zero out the blocks $tB_{i+1,j}, \ldots, tB_{l,i}$ left of them, requiring operations over polynomials.
The pencil $tB_{l+1,l+1} - A_{l+1,l+1}$, which contains the finite structure of the pencil, can then be diagonalized with its invariant polynomials as diagonal entries with the Smith algorithm. Having kept track of the change of basis matrices $Q(t)$ and $R(t)$ required to put $tB - A$ into block diagonal form, we find our singular eigenvectors as the columns of $R(t)$ corresponding to zero columns of the block diagonal pencil. Non-singular eigenvectors for $t_0 \in \mathbb{F}$ are obtained by evaluating at t_0 the columns of $R(t)$ corresponding to an invariant polynomial of the pencil with t_0 as root. Generalized eigenvectors are obtained by differentiating these columns over the ring $\mathbb{Z}$ an appropriate number of times depending on the dimension of the generalized eigenspace.
In this way the spaces $E_t(\varphi_i, \psi_i)$ and $E(\varphi_i, \psi_i)$ are built for every i.

Numerical experiments

The following figure shows the result of computations of persistent vectors in H_1 homology over $\mathbb{Z}_{19}$ with our algorithm. The left picture shows the persistence barcodes for the eigenvector for eigenvalue 2, where we apply the map $z \mapsto z^2$ to a sample of 100 random points on $S^1 \subset \mathbb{C}$ subject to Gaussian noise with a standard deviation varying from 0 to 0.3. The bar is drawn blue where the vector is only associated with 2, and red where it is actually a singular vector. It can be seen that as the noise level increases, the persistence of the vector decreases, and it becomes singular for a longer time.
The right picture shows persistence diagrams for generalized eigenvectors associated with eigenvalue 1 and for singular eigenvectors, where we apply the map $(x, y) \mapsto (x + y, y)$ to 200 random points on the torus obtained by identifying the left and right, and top and bottom, edges of $[0, 1] \times [0, 1]$. To make clearer the actual dynamics of the map, the vectors in $e_{1,i}(E(\varphi_i, \psi_i))$ are quotiented out at every step when computing vectors for $t = 1$.





References

1. F.R. Gantmacher, *The Theory of Matrices*, Chelsea Publishing Company, New York, 1959, 374+276 p.
2. P. van Dooren, The Computation of Kronecker's Canonical Form of a Singular Pencil, Lin. Alg. Appl., 27 (1979), 103–140.
3. H. Edelsbrunner, G. Jabłoński, M. Mrozek, The Persistent Homology of a Self-Map, Found. Comp. Math., DOI=10.1007/s10208-014-9223-y (2014), 32 p.

Institute of Computer Science and Computational Mathematics, Jagiellonian University, Poland
Mail: ethier@ii.uj.edu.pl