Discrete Morse Functions obtained from Knots

Moshe Cohen
Technion - Israel Institute of Technology

The perspective from Discrete Morse Theory as in [4]

Let Δ be any 2-complex of the 2-sphere. This complex need not be simplicial. Multiple edges and loops are allowed.

Let G be the graph given by the 1-skeleton of this complex, realized by a plane embedding.

Let Δ^* be the dual block complex with its 1-skeleton given as the graph G^* realized by a plane embedding dual to the graph G.

Let $\mathcal{F}(\Delta)$ be the face poset of Δ, viewed as a plane tripartite graph $\widehat{\Gamma}$ with the same embeddings as above.

Let v_0 and f_0 be the two critical cells of the 2-sphere of dimension 0 and 2, respectively.

Assumption. The critical cells v_0 and f_0 must form a square face in $\widehat{\Gamma}$, the face poset $\mathcal{F}(\Delta)$.

The perspective from Knot Theory as in [1, 2, 3]

The graph G is the *Tait checkerboard graph*, with its plane dual G^* the other Tait checkerboard graph.

The graph $\widehat{\Gamma}$ is the *overlaid Tait graph*, obtained by overlaying G and G^*.

Delete two vertices on the same square face to obtain Γ, the *balanced overlaid Tait graph*.

Equivalent to spanning tree models on G, perfect matching models on Γ can be used to compute knot invariants.

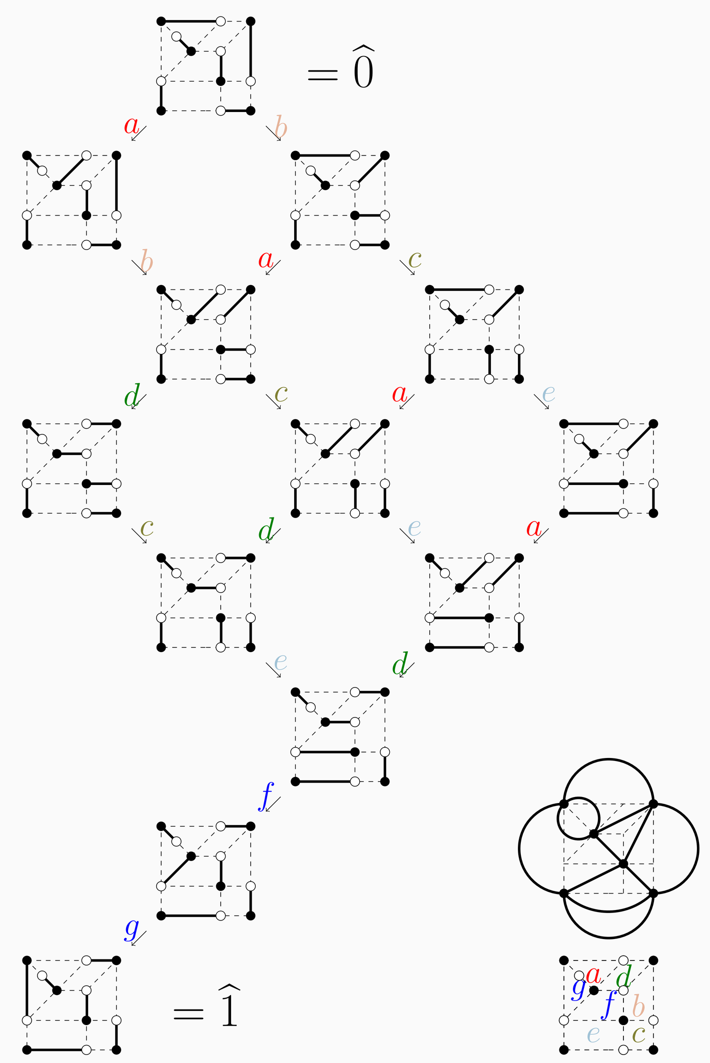
Main Result

Discrete Morse functions on the 2-complex Δ of the 2-sphere correspond to perfect matchings on Γ constructed from a knot diagram.

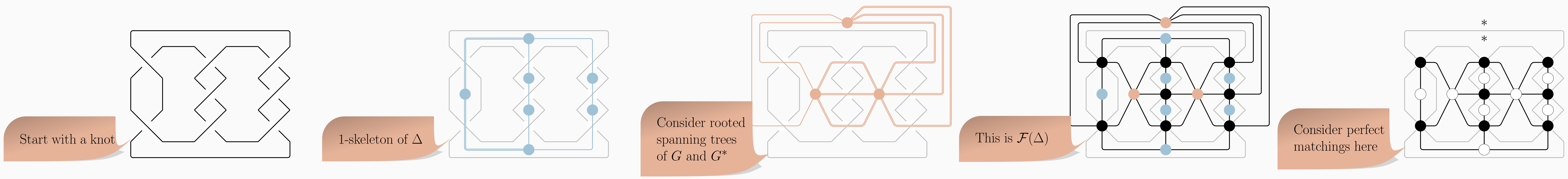
Applications in Knot Theory

- the Alexander polynomial [1, C-Dasbach-Russell]
- the Jones polynomial of a pretzel knot [2, C-]
- the twisted Alexander polynomial of a knot together with a representation (using p-lifts) [1, C-Dasbach-Russell]
- the δ-graded knot homology theories [C- unpublished]
- [Koseleff-Pecker] show every knot has a Γ that is a grid graph (also see [5, C-])
- [Huggett-Mofatt-Virdee] use $\widehat{\Gamma}$ to study ribbon graphs from cables
- [Kravchenko-Polyak] use Γ obtained on a torus to study cluster algebras
- [Kidwell-Luse] study "one-spinners" generalizing Abe's clock number

Lattice of dMf's [3]



Various graphs obtained from a knot diagram



The knot 8_{19} as the $(-2, 3, 3)$-pretzel knot, the corresponding signed Tait graph G, the dual signed Tait graph G^*, the overlaid Tait graph $\widehat{\Gamma}$ (all faces are square), and the balanced overlaid Tait graph Γ.

Future Work

In an upcoming project with several co-authors, we consider so-called *links on thickened surfaces*, the natural generalization to higher genus.

Instead of the Tait checkerboard graph, this involves the *all-A ribbon graph*, a graph on a surface, associated to the knot diagram.

The associated objects are not exactly discrete Morse functions but are related.

Question. What role does the assumption play?

Question. Could a higher-dimensional analog extend to 2-knots?

References

[1] Moshe Cohen, Oliver T. Dasbach, and Heather M. Russell, *A twisted dimer model for knots.*, Fundam. Math. **225** (2014), no. 1, 57–74.

[2] Moshe Cohen, *A determinant formula for the Jones polynomial of pretzel knots.*, J. Knot Theory Ramifications **21** (2012), no. 6, 23pp.

[3] Moshe Cohen and Mina Teicher, *Kauffman's clock lattice as a graph of perfect matchings: a formula for its height*, arXiv:1211.2558, 2012.

[4] Moshe Cohen, *A correspondence between complexes and knots*, arXiv:1211.2553, 2012.

[5] Moshe Cohen, *The Jones polynomials of 3-bridge knots via Chebyshev knots and billiard table diagrams*, arxiv:1409.6614, 2014

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