Get a Better Description of your Data through a New Idea on Persistence

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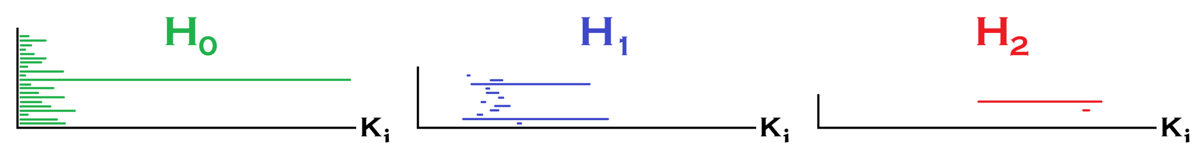
1. Introduction

Persistent Homology is a powerful tool to study *The Shape of Data*. Here is an overview of A_∞-**Persistence**, a refinement that allows you to find much more about that *shape*.

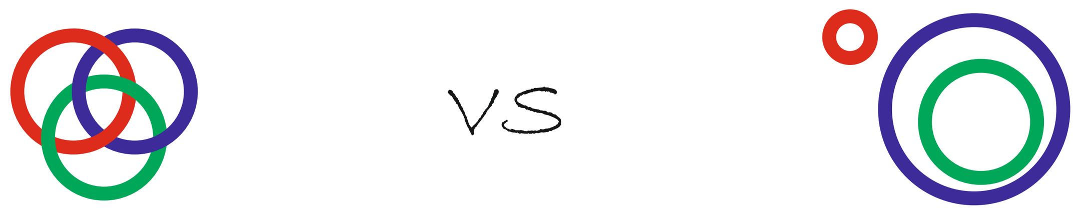
2. Some Motivation

Imagine that the following are the only non-trivial barcodes we get after running a Persistent Homology algorithm:



How can we know if the underlying space is then similar to a Torus or to a wedge of spheres $\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1$?

Betti numbers cannot distinguish between them, nor between other pairs of spaces like Borromean rings and unlinked circumferences,



but there are **operations in (co)homology** which **can tell them apart**.

3. The Big Picture

Think of a **point cloud data set** as a sampling of an unknown topological space X.

In this context, Persistence builds a sequence of nested spaces

$$\mathcal{K}: \quad K_0 \hookrightarrow K_1 \hookrightarrow \ldots \hookrightarrow K_N$$

so that when you study how the topology changes along the sequence, you get information on the topology of X, and ultimately, on the structure of the original data set.

If we **work over a field**, Persistent Homology can compute $H_*(X)$, whereas A_∞**-Persistence** can describe X at a much deeper level; namely, for instance, that of

- **linking number** and **Milnor invariants**, when defined,

  or

- **cup product** and **generalized Massey products**.

To give a brief idea on how A_∞-Persistence works, in this poster **we focus on** the structural operations Δ_n of an A_∞-coalgebra on Homology **(see box 4)**.

4. A_∞-coalgebra on $H_*(Y)$

An **A_∞-coalgebra structure** on a graded module M consists of a sequence of operations

$$\Delta_n : M \longrightarrow M^{\otimes n},$$

one for each $n \geq 1$, satisfying certain compatibility relations. For low values of n, these relations look like this: Δ_1 and Δ_2 can be seen, respectively, as a differential and a coproduct making (M, Δ_1, Δ_2) a differential graded **coalgebra**, coassociative **up to** the chain homotopy Δ_3.

The homology $H_*(Y)$ of any topological space Y is endowed with a **standard A_∞-coalgebra** structure for which Δ_2 is the induced map in homology by any approximation to the diagonal. The analogue holds for cohomology too, and these structures carry **more information than that of the cohomology algebra**.

Concepts and Results

Notation for Boxes 5 & 6: Use notation in Box 3 and for all $p \geq 0$, $0 \leq i \leq j \leq N$ and $n \geq 1$, let us denote by

$$f_p^{i,j} : H_p(K_i) \longrightarrow H_p(K_j) \qquad \text{and} \qquad \Delta_n^j : H_*(K_j) \longrightarrow \left(H_*(K_j)\right)^{\otimes n}$$

the morphism induced in homology by the inclusion $K_i \hookrightarrow K_j$, and the operation Δ_n of the standard A_∞-coalgebra on $H_*(K_j)$, respectively.

5. Classical Persistent Homology

Definitions

p^{th} ***persistent group*** between K_i and K_j: $\mathcal{H}_p^{i,j}(\mathcal{K}) = \text{Im } f_p^{i,j}$.

A homology class $\alpha \in H_p(K_i)$:

- is ***alive*** at K_j if $f^{i,j}\alpha \in H_p(K_j) - \{0\}$
- is ***born*** at K_i if α is alive at K_i and $\alpha \notin \text{Im } f^{i-1,i}$
- ***dies*** at K_j if α is alive at K_{j-1} but not at K_j

Main result

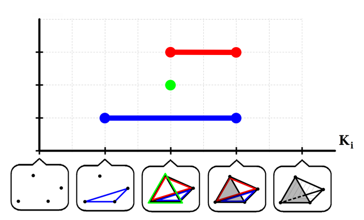
Theorem. *[1] Given $p \geq 0$, there exists a basis $\mathcal{B}$ of $H(\mathcal{K}) = \oplus_{i=1}^N H_*(K_i)$ such that, for any $1 \leq i \leq j \leq N$,*

$$\dim \mathcal{H}_p^{i,j}(\mathcal{K}) = \#\{\beta \in \mathcal{B} \cap H_p(K_i),\ \beta \text{ is alive at } K_j\}.$$

This information can be set in a ***Barcode***, where

$$\dim \mathcal{H}_p^{i,j}(\mathcal{K}) = \#\{\text{bars containing } [i,j]\},$$

where **long bars** correspond to p-homology classes in X. Here is an example of Barcode for $p = 1$:



6. A_∞-persistence

Definitions

p^{th} ***Δ_n-persistent group*** between K_i and K_j: $(\Delta_n)_p^{i,j}(\mathcal{K}) = \text{Im } f_p^{i,j}|_{\cap_{k=i}^j \text{Ker}(\Delta_n^k \circ f_p^{i,k})}$.

Remark. *Since Δ_1^i is the differential in $H_*(K_i)$, taken to be 0, we can recover classical persistence as $(\Delta_1)_p^{i,j}(\mathcal{K}) = \text{Im } f_p^{i,j} = H_p^{i,j}(\mathcal{K})$.*

A homology class $\alpha \in H_p(K_i)$:

- is ***Δ_n-awake*** at K_j if $f^{i,j}\alpha \in \ker \Delta_n^j - \{0\}$
- ***Δ_n-wakes up*** at K_i if α is Δ_n-awake at K_i and $\alpha \notin \text{Im } f^{i-1,i}|_{\ker \Delta_n^{i-1}}$
- ***Δ_n-falls asleep*** at K_j if α is Δ_n-awake at K_{j-1} but not at K_j

Remark. *Observe that, as $\Delta_1^i = 0$ for any $H_*(K_i)$, the concepts of Δ_1-awakening and Δ_1-falling asleep coincide with those of birth and death in classical persistence.*

Main results

Theorem. *In general, a non zero-class α can Δ_n-wake up and Δ_n-fall asleep several times along $\mathcal{K}$.*

Theorem. *Given $n \geq 1$, and $p \geq 0$, there exists a basis $\mathcal{B}$ of $H(\mathcal{K}) = \oplus_{i=1}^N H_*(K_i)$ such that, for any $1 \leq i \leq j \leq N$,*

$$\dim(\Delta_n)_p^{i,j}(\mathcal{K}) = \#\{\beta \in \mathcal{B} \cap H_p(K_i),\ \beta \text{ is } \Delta_n\text{-awake at } K_k \text{ for } k = i, \ldots, j\}.$$

We can also encode this information in a ***Barcode*** so that

$$\dim (\Delta_n)_p^{i,j}(\mathcal{K}) = \#\{\text{bars containing } [i,j]\},$$

where **long bars** correspond to p-homology classes α that are in $\ker \Delta_n$ in the space X underlying the point cloud.

7. Conclusions

In the literature, **fixing some i**, the cup product in $H^*(K_i)$ is used to study cohomology classes that are alive at K_i.

With our methods, **we do not need to focus on the cup product of a chosen K_i**; instead, dualizing the case $n = 2$ in Box 6 with some modifications yields to an **actual persistent approach to the cup product** that uses all the ring structures in $\{(H^*(K_i), \smile)\}_i$ at the same time.

Furthermore, we can apply this to **generalized Massey products** as well, which amounts to **much more information than** the one **classical Persistence** gives.

Also, thanks to the computational approach to A_∞-structures by Pedro Real et ál. and to the decomposition algorithms of Zigzag Persistence, A_∞-persistence can be **explicitly computed.**

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Reference of Theorem in Box 5: [1] G. Carlsson and A. Zomorodian, Computing persistent homology, *Discrete and Computational Geometry*, **33**(2) (2005), 249–274.