The Morse Theory of Čech and Delaunay Complexes

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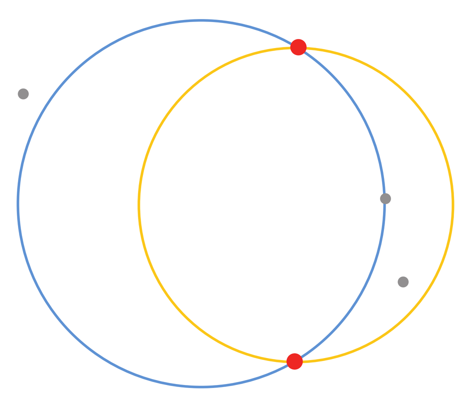
Čech and Delaunay functions

$X \subset \mathbb{R}^d$: finite point set (in general position)

Simplices $\Delta(X)$: nonempty subsets

Two functions on simplices $Q \subseteq X$:

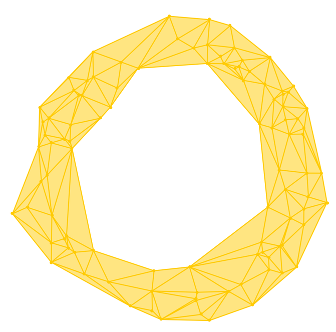
- **Čech function** $f_C(Q)$:
  radius of smallest enclosing sphere of Q
- **Delaunay function** $f_D(Q)$:
  radius of smallest empty circumsphere of Q
  - defined only if Q has an empty circumsphere: $Q \in \mathrm{Del}\, X$



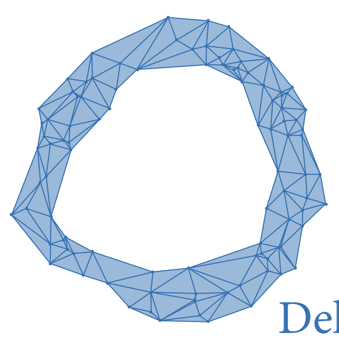
Čech and Delaunay complexes

Define for any radius r:

- **Čech complex** $\mathrm{Cech}_r = f_C^{-1}(-\infty, r]$
  - all simplices having an enclosing sphere of radius $\leq r$
- **Delaunay–Čech complex** $\mathrm{DelCech}_r = \mathrm{Cech}_r \cap \mathrm{Del}$
  - restriction of Čech complex to Delaunay simplices
- **Delaunay complex** (α-shape for $\alpha = r$) $\mathrm{Del}_r = f_D^{-1}(-\infty, r]$
  - all simplices having an empty circumsphere of radius $\leq r$



$\mathrm{DelCech}_r$



Del_r

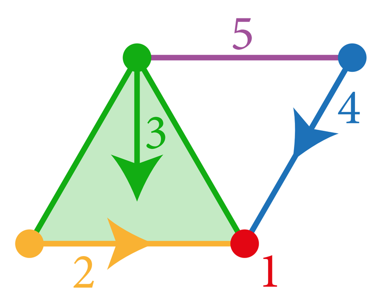
We know that $\mathrm{Cech}_r(X) \simeq \mathrm{Del}_r(X) \simeq \mathrm{B}_r(X)$ (Borsuk 1947).
But we also have $\mathrm{Del}_r(X) \subseteq \mathrm{DelCech}_r(X) \subseteq \mathrm{Cech}_r(X)$.

- Are the three complexes connected by simplicial collapses?
- For which r do they change their homotopy type?

Generalized discrete Morse theory

A function $f : K \to \mathbb{R}$ on a simplicial complex is a **generalized discrete Morse function** if for all $t \in \mathbb{R}$:

- the sublevel set $K_t = f^{-1}(-\infty, t]$ is a subcomplex
- the level set $f^{-1}(t)$ is an interval of the face poset:
  $[L, U] = \{Q : L \subseteq Q \subseteq U\}$.



If $f^{-1}(t) = \{Q\}$ then t is a **critical value** (and Q a **critical simplex**).

Theorem (Forman 1998)
If $(s, t]$ contains no critical value of f, then $K_t \searrow K_s$.

Morse theory of Čech and Delaunay functions

Theorem
The Čech and Delaunay functions are generalized discrete Morse functions.
The critical simplices of both are the Delaunay simplices Q with $f_D(Q) = f_C(Q)$.
These are precisely the Delaunay simplices containing the circumcenter in their interior.

Sphere minimization problems

Both Čech and Delaunay function are defined using smallest spheres satisfying certain constraints:

$$\begin{aligned} &\underset{r,z}{\text{minimize}} \quad r \\ &\text{subject to} \ \|z - q\| \leq r, \ q \in Q, \\ &\qquad\qquad\ \|z - e\| \geq r, \ \ e \in E. \end{aligned}$$

Here r is the radius and z is the center of the sphere.

- Čech function: choose $E = \varnothing$
- Delaunay function: choose $E = X$

Selective Delaunay complexes

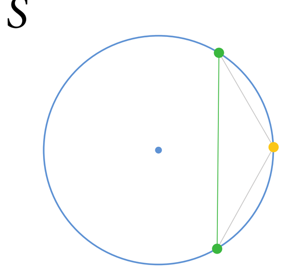
Define for any subset $E \subseteq X$:

- E**-Delaunay function** $f_E(Q)$:
  radius of smallest E-empty enclosing sphere of Q
  - defined only if Q has an E-empty enclosing sphere: $Q \in \mathrm{Del}(X, E)$
- E**-Delaunay complex** $\mathrm{Del}_r(X, E) = f_D^{-1}(-\infty, r]$

Čech and Delaunay intervals

Consider a sphere S enclosing the points $Q \subseteq X$ and excluding the points $E \subseteq X$

- Let encl S be all points of X enclosed by S, and excl S excluded by S
- Let on S be the points of X on S
- Write the center of S as an affine combination $z_S = \sum_{x \in \text{on}\, S} \mu_x x$
- Let $\text{front}\, S = \{x \in \text{on}\, S \mid \mu_x > 0\}$, and $\text{back}\, S = \{x \in \text{on}\, S \mid \mu_x < 0\}$

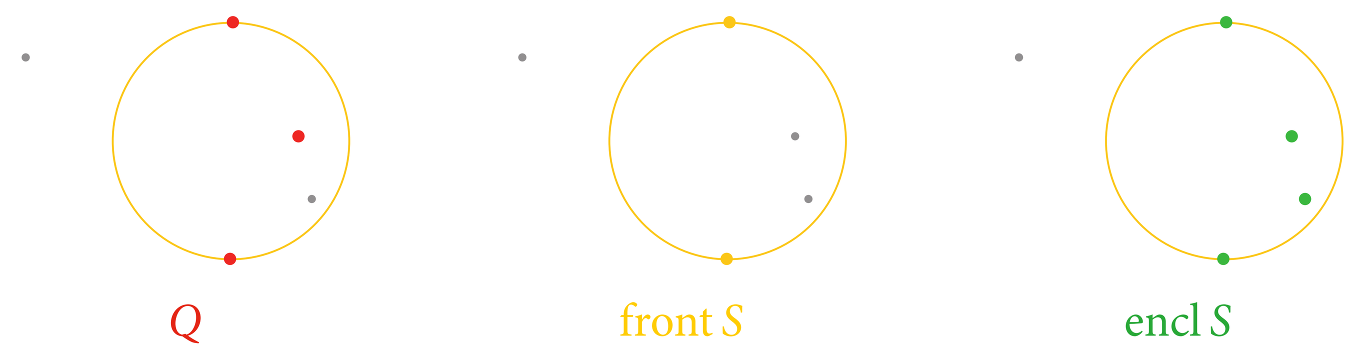


The **Karush–Kuhn–Tucker** conditions for the sphere minimization problem yield:

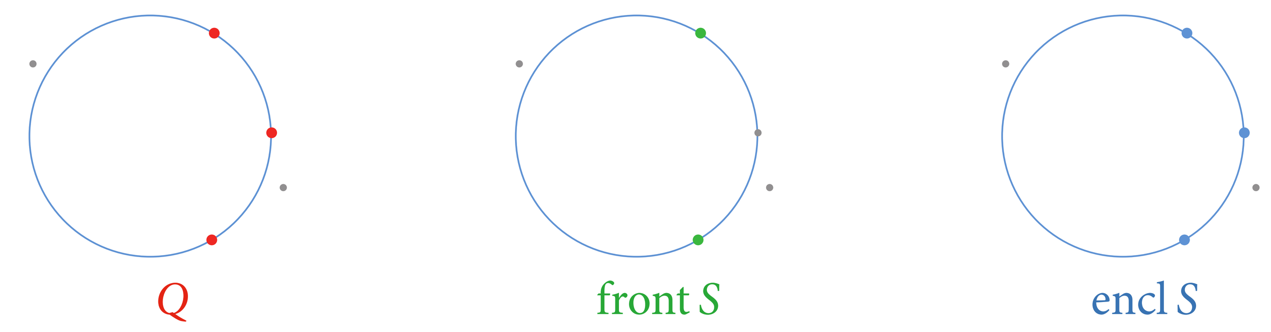
Lemma
A sphere S enclosing Q and excluding E is the smallest such sphere iff:

- $z_S \in \mathrm{aff}(\text{on}\, S)$
- $Q \in [\text{front}\, S, \text{encl}\, S]$ *(i.e., $\text{front}\, S \subseteq Q \subseteq \text{encl}\, S$)*
- $E \in [\text{back}\, S, \text{excl}\, S]$

The Čech intervals are the level sets $f_C^{-1}(t)$ of the Čech function:



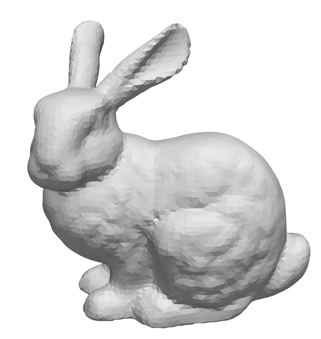
The Delaunay intervals are the level sets $f_D^{-1}(t)$ of the Delaunay function:



Theorem
The intersection of non-critical Čech and Delaunay intervals is again non-critical.

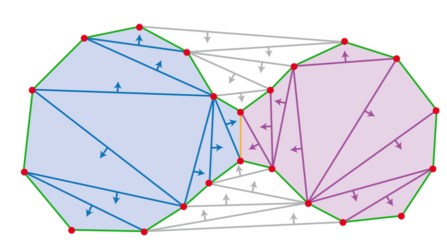
Wrap complexes

Generalizes and greatly simplifies the surface reconstruction method **Wrap** (Edelsbrunner 2003)



Define for any radius r:

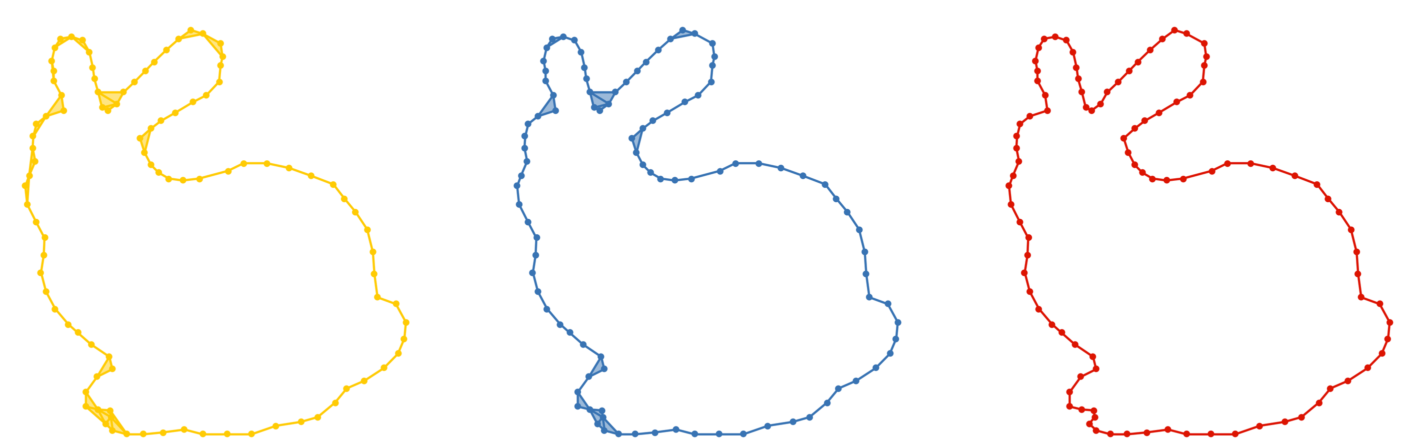
- **Wrap complex** $\mathrm{Wrap}_r = \bigcup \downarrow \mathrm{Crit}_r$
  - Crit_r denotes the critical simplices with value $\leq r$
  - $\downarrow$ denotes the descending set of Delaunay intervals (with the partial order induced by the face relation)
  - Intuition from smooth Morse theory: union of descending manifolds for all critical points with value $\leq r$



A sequence of collapses

Theorem
Let $E \subseteq E' \subseteq X$. Then $\mathrm{Del}_r(X, E) \searrow \mathrm{Del}_r(X, E) \cap \mathrm{Del}\,(X, E') \searrow \mathrm{Del}_r(X, E')$.

Corollary
For any given radius r, Čech, Delaunay–Čech, Delaunay, and Wrap complexes are simple-homotopy equivalent. In particular, $\mathrm{Cech}_r \searrow \mathrm{DelCech}_r \searrow \mathrm{Del}_r \searrow \mathrm{Wrap}_r$.



References

[1] H. Edelsbrunner. *Surface reconstruction by wrapping finite sets in space*, 2003.

[2] R. Forman. *Morse theory for cell complexes*, 1998.

[3] R. Freij. *Equivariant discrete Morse theory*, 2009.