Structure of the Iwahori-Hecke Algebra

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Notation

F - Non-archimedean local field.
$\mathcal{O}$ - ring of integers, $\mathfrak{p} = (\pi)$ - maximal ideal.
$k := \mathcal{O}/\mathfrak{p}$, $|k| = q$.

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G - Split, connected, reductive group defined over F.
A - split maximal torus A.
$B = AN$ - Borel subgroup containing A.
We assume that G, A, N etc. are defined over $\mathcal{O}$.

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$W := N_{G(F)}(A(F))/A(F)$ - the finite Weyl group.

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We will sometimes write G, A for $G(F), A(F)$ (respectively) for the sake of brevity.

The Iwahori subgroup and the Iwahori Hecke Algebra

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The Iwahori subgroup I *of* K *is the inverse image of* $B(k)$ *in* $G(\mathcal{O})$ *under the map* $G(\mathcal{O}) \to G(k)$.

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Define the Iwahori-Hecke algebra,

$$\mathcal{H}_I := \mathrm{C}_c(I \setminus G(F)/I).$$

Finite dimensional Hecke algebra

$$\mathcal{H}_f := \mathrm{C}(B(k) \setminus G(k)/B(k)).$$

By Bruhat decomposition,

$$\{T_w := 1_{B(k)wB(k)} | (w \in W)\}$$

forms a basis of $\mathcal{H}_f$.

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Proposition

For $f \in \mathrm{C}(B(k) \setminus G(k)/B(k))$, we can define $\tilde{f} \in \mathrm{C}_c(I \setminus K/I)$ in a natural way. The map $f \mapsto \tilde{f}$ gives a $\mathbb{C}$-algebra isomorphism

$$\mathrm{C}(B(k) \setminus G(k)/B(k)) \cong \mathrm{C}_c(I \setminus K/I).$$

Theorem (Iwahori)

$\mathcal{H}_f$ is a free $\mathbb{C}$-algebra on generators T_{s_i} (indexed by the set S of simple reflections of W) subject to the following relations:

- $T_{s_i}^2 = q + (q-1)T_{s_i}$
- $T_{s_i} \cdot T_{s_j} \cdot T_{s_i} \ldots = T_{s_j} \cdot T_{s_i} \cdot T_{s_j} \ldots \quad (i \neq j)$
  with m_{ij} terms on both sides, with $m_{ij} \geq 2$, same as those occuring in the braid relations in the presentation of the coxeter group W in terms of S.

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$\mathcal{H}_f$ is very similar in structure to $\mathbb{C}[W]$.

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Proposition (Iwahori and Matsumoto)

$G = \sqcup_{x \in \tilde{W}_a} IxI.$

Thus the elements $\{T_x := 1_{IxI},\ x \in \tilde{W}_a\}$ form a natural basis of $\mathcal{H}_I$.

The extended affine Weyl group for $\mathrm{GL}_n(F)$

Let s_i $(1 \leq i \leq n-1)$ be the permutation matrix in $\mathrm{GL}_n(F)$ corresponding to the transposition which interchanges e_i and e_{i+1}. Further let

$$s_0 = \begin{pmatrix} 0 & & & & & & \pi^{-1} \\ & 1 & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & 1 & \\ \pi & & & & & & 0 \end{pmatrix}, \ t = \begin{pmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & 1 & & & \\ & & & 0 & 1 & & \\ & & & & 0 & 1 & \\ & & & & & 0 & 1 \\ \pi & & & & & & \end{pmatrix}$$

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Note that $ts_it^{-1} = s_{i-1}$.

$$\tilde{W}_a =< s_0, ..., s_{n-1}, t >$$

with the following relations:

- $s_i^2 = 1,\ 0 \leq i \leq n-1,$
- $(s_i s_j)^{m_{ij}} = 1$ where $m_{i,i+1} = 3$ and $m_{i,j} = 2$ if $|i-j| \mod n > 1,$
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Thus $\tilde{W}_a$ is almost like S_n.

Let length function, ℓ is defined to be the minimum number of generators of the type s_i (i.e. excluding any occurrences of the generator t). Note that ℓ restricted to W is the usual length function.

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It can be easily checked that $\tilde{W}_a \cong S_n \ltimes \mathbb{Z}^n$ where $\mathbb{Z}^n$ is embedded as the diagonal matrices with powers of π.

The Iwahori-Matsumoto presentation for $G = \mathrm{GL}_n(F)$

Theorem

$\mathcal{H}_I$ is an algebra generated by appropriately indexed T_{s_i} and T_t such that the following relations are satisfied:

- $T_{s_i} \cdot T_{s_i} = q + (q-1)T_{s_i}$.
- $T_w \cdot T_{w'} = T_{ww'}$ if $\ell(ww') = \ell(w) + \ell(w')$.
- $T_{s_{i-1}} \cdot T_t = T_t \cdot T_{s_i}$.

(Here $i+1$ is to be interpreted as 0 if $i = n-1$.)

The co-character lattice

$$X_*(A) := \mathrm{Hom}(\mathbb{G}_m, A(F))$$
$$\pi^\mu := \mu(\pi), \quad \forall \mu \in X_*(A)$$

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In case of $\mathrm{GL}_n(F)$,

$$\mu \in X_*(A) \leftrightarrow \mathrm{diag}(\pi^{t_1}, ..., \pi^{t_n}) \leftrightarrow (t_1, ..., t_n)$$

where $(t_1, ..., t_n) \in \mathbb{Z}$.

Definition

Call an element $\mu \in X_*(A)$ dominant if

$$\pi^{\mu}(I \cap N)\pi^{-\mu} \subset I \cap N.$$

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It follows that $\pi^{-\mu}(I \cap \bar{N})\pi^{\mu} \subset I \cap \bar{N}$.

The toric subalgebra

Let $\mathcal{B} = \mathbb{C}[X_*(A)]$. Using the description of $X_*(A)$,

$$\mathcal{B} \cong \mathbb{C}[X_1, ..., X_n, X_1^{-1}, ..., X_n^{-1}].$$

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For $\mu \in X_*(A)$ dominant define

$$\Theta_\mu = \delta^{1/2}(\pi^\mu) T_{\pi^\mu},$$

where δ is the modulus function of $B(F)$.

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Proposition

Θ_μ is invertible in $\mathcal{H}_I$.

Let μ and η are dominant. Then,

$$\begin{aligned} I\pi^{\mu}I\pi^{\eta}I &= I\pi^{\mu}(I \cap N)(I \cap A)(I \cap \bar{N})\pi^{\eta}I \\ &\subset I(I \cap N)\pi^{\mu}(I \cap A)\pi^{\eta}(I \cap \bar{N})I \\ &\subset I\pi^{\mu}\pi^{\eta}I \end{aligned}$$

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So we have, $I\pi^{\mu} I \pi^{\eta} I = I\pi^{\mu+\eta} I$, which implies

$$\Theta_{\mu} * \Theta_{\eta} = \Theta_{\mu+\eta}.$$

For $\mu \in X_*(A)$, write $\mu = \mu^+ - \mu^-$, where μ^+, μ^- are dominant. So for an arbitrary μ define

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$\Theta(\mathcal{B})$ is an abelian subalgebra of $\mathcal{H}_I$ and is known as the toric subalgebra.

The Bernstein relations

We have the following commutation relation.

Proposition

Let $\mu \in X_*(A)$, $s \in S$ and α the simple root corresponding to s. Then

$$T_s * \Theta_\mu = \Theta_{s\mu} * T_s + (q-1)\frac{\Theta_\mu - \Theta_{s\mu}}{1 - \Theta_{-\check{\alpha}}}.$$

($s\mu = \mu - \langle \alpha, \mu \rangle \check{\alpha}$)

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Proposition

The elements $\Theta_\mu T_w \in \mathcal{H}_I$ form a basis over $\mathbb{C}$.

The Bernstein presentation

Theorem

Let $\mathcal{H}$ be the algebra with the generators

$$\{T_w, w \in W,\ \Theta_\mu, \mu \in X_*(A)\}$$

such that the following relations are satisfied:

- $T_s \cdot T_s = q + (q-1)T_s\ \forall s \in S.$
- $T_w \cdot T_{w'} = T_{ww'}$ if $\ell(ww') = \ell(w) + \ell(w').$
- $\Theta_\mu \cdot \Theta_\eta = \Theta_{\mu+\eta}.$
- *For $s = s_\alpha \in S, \mu \in \mathcal{B}$ we have,*

$$T_s \cdot \Theta_\mu = \Theta_{s\mu} \cdot T_s + (q-1)\frac{\Theta_\mu - \Theta_{s\mu}}{1 - \Theta_{-\check{\alpha}}}.$$

Then $\mathcal{H} \cong \mathcal{H}_I$.

Description of the center

W acts naturally on $\mathcal{B}$. It follows easily from the commutation relation that

$$\Theta(\mathcal{B}^W) = \mathrm{Span} < \sum_{w \in W} \Theta_{w(\mu)} > \ \subset Z(\mathcal{H}_I).$$

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Theorem (Bernstein)

$\Theta : \mathcal{B}^W \xrightarrow{\sim} Z(\mathcal{H}_I)$.

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Theorem (Bernstein)

$\Theta : \mathcal{B}^W \overset{\sim}{\to} Z(\mathcal{H}_I)$.

It is easy to check that $\mathcal{H}_I$ is a f.g. module over $\mathcal{B}$ and $\mathcal{B}$ over $\mathcal{B}^W$. Thus $(\mathcal{H}_I)$ is f.g. over $Z(\mathcal{H}_I)$ and hence is noetherian.

Summary

- Structure of $\mathcal{H}_I$ is goverened by $\tilde{W}_a$.
- $\mathcal{H}_I \cong \mathcal{H}_f \otimes_{\mathbb{C}} \mathcal{B}$ (as $\mathbb{C}$-vector spaces). ($\mathcal{B}$ abelian)
- $T_s \cdot \Theta_\mu = \Theta_{s\mu} \cdot T_s + (q-1)\frac{\Theta_\mu - \Theta_{s\mu}}{1-\Theta_{-\check{\alpha}}}$.
- $Z(\mathcal{H}_I) \cong \mathcal{B}^W$.

Thank you for your attention.