A PLAN FOR LECTURES ON HIGHER STRING STRING OPERATIONS

A.P.M. KUPERS

Contents

This note contains the plan and references for a pair of two-hour lectures about the construction of higher string operations using radial slit configurations, based on [Kup11]. The first lecture will discuss the model for moduli space that we will be using and explains how to construct the string operation for cobordisms corresponding to particularly easy to handle points in the moduli space. The second lecture discusses some of the intricracies of globalizing the construction of the string operations, in a way that slightly diverges from [Kup11], but generalizes to a partial compactification of moduli space. We will also talk about applications of this partial compactification.

1. Lecture 1: Radial slit configuration and the construction of the operation associated to a single generic configuration

1.1. **Radial slit configurations.** If Σ is a two-dimensional cobordism of any genus, but with at least one incoming and one outgoing boundary component, then Bödigheimer gives a model for the moduli space of cobordisms with parametrized incoming and outgoing boundary components isomorphic to Σ. This is the model of radial slit configurations. A full discussion of it can be found in [Böd06] and an abbreviated one in [Kup11, sections 1 & 2].

The lecture should discuss the following features of the radial slit configuration model:

(i) The definition of the space of radial slit configurations Rad_Σ, by first defining possibly degenerate preconfigurations, then identifying preconfigurations giving rise to the same surface and finally removing the degenerate configurations. This is [Böd06, sections 3, 4 & 7], though he doesn't parametrize the outgoing boundary until section 9. Also mention the harmonic compactification as in [Böd06, section 8]. There is a different construction glueing open subspaces of the bar complexes of symmetric groups. For parallel slit configurations, this can be found in [ABE08].
(ii) A description of composition and disjoint union of cobordisms in the radial slit configuration model for moduli space. This is [Böd06, section 9] and [Kup11, section 1.4].
(iii) The construction of the universal surface and ribbon graph bundles over Rad_Σ. The surface bundle is done implicitly in [Böd06, section 4] and the graph bundle is Γ of [Kup11, section 2.4].

Giving many examples is strongly recommended! Also, the discussion should include the definition of a generic radial slit configuration. This is a radial slit configuration such that none of the slits are on the same radial segment. For this class of radial slit configurations the third part of this lecture will construct string operations.

Date: Januari 27th 2012.

1.2. **Umkehr maps via the Pontryagin-Thom construction and propagating flows.** Since string operations in general arise via a push-pull construction, we need a description of umkehr maps[1]. Our preferred model is the Pontryagin-Thom construction. For maps $X \to Y$ of spaces that admit a tubular neighborhood with orientation in a suitable sense, this gives a wrong-way map in homology that shifts degree. For an embedding of oriented manifolds this construction is straightforward. In the case of mapping spaces into manifolds, propagating flows provide a good explicit model for lifting the tubular neighborhood needed to do the Pontryagin-Thom construction.

One can make a coherent story out of the following topics (see [Kup13, chapter 2]).

(i) The Pontryagin-Thom construction, focussing on the case of the intersection product on $H_*(M)$. Cohen-Voronov does this [CV06, section 1.1], as does Cohen-Jones [CJ02, section 1].
(ii) Fill in the details for mapping spaces into manifolds by explaining propagating flows and showing how to use them to lift tubular neighborhoods from the manifolds, thus allowing one to extend the Pontryagin-Thom construction to mapping spaces into manifolds, as in [Kup13, section 2.1.4]. Also look at [CV06, section 1.2] and [God07, section 3.1].

1.3. **The operation associated to a generic configuration.** The final part of the lecture will combine the first and second parts to give a construction of the string operation associated to a single generic configuration. Though of course implicit in [Kup11], I will write a note explicitly doing this construction. It might be interesting to take a look at the construction in [God07, section 3.1] for comparison.

2. Lecture 2: Higher string operations and compactification

2.1. **Globally constructing the string operations.** The third part of previous lecture discussed how to construct a string operation associated to a single generic configuration. As a follow-up we explain how to do the construction in general, in a continuous fashion over the model Rad_Σ for moduli space discussed in the previous lecture. The idea is to do the construction over the space of preconfigurations first, in such a way that it is compatible with the identifications. There are several ways for doing this. One is given in [Kup11], but in the lecture we will give a second construction for the benefit of the third part of this lecture. We will need a bit of the technology of parametrized spectra, as in [MS06].

2.2. **Compatibility with composition and disjoint union.** To complete the construction of the higher string operations one needs to prove that these satisfy the conditions of the homological field theory. This algebraic structure was defined in [God07, section 4.1]. The second part of the lecture outlines that argument. This is an expanded version of [Kup11, section 4].

2.3. **Partial compactification and its applications.** Finally, we look towards some of the other topics of the workshop. To do this, we discuss how the construction in the first part naturally extends to a partial compactification of the radial slit configurations. In this framework we can describe the Goresky-Higston higher cobrackets and some vanishing results (including Tamanoi's result [Tam09]). This is unpublished, but notes will appear before the workshop.

References

[ABE08] J. Abhau, C.F. Bödigheimer, and R. Ehrenfried, *Homology computations for mapping class groups and moduli spaces of surfaces with boundary*, Heiner Zieschang Gedenkschrift (M. Boileau, M. Scharlemann, and R. Weidmann, eds.), Geometry and Topology Monographs, vol. 14, 2008, available at `http://arxiv.org/pdf/0712.4254`, pp. 1–25.

[Böd06] C.F. Bödigheimer, *Configuration models for moduli space of Riemann surfaces with boundary*, Abh. Math. Sem. Univ. Hamburg (2006), no. 76, 191–233, available at `http://www.math.uni-bonn.de/people/cfb/PUBLICATIONS/radial-moduli.pdf`.

[CJ02] R.L. Cohen and J.D.S. Jones, *A homotopy theoretic realization of string topology*, Math. Ann. **324** (2002), no. 4, 773798, available at `http://arxiv.org/pdf/math/0107187v2`.

[1] A nice overview of various constructions and their general properties is [CM12, section 2], though unfortunately it doesn't discuss the Pontryagin-Thom construction.

[CM12] David Chataur and Luc Menichi, *String topology of classifying spaces*, J. Reine Angew. Math. **669** (2012), 1–45, available at http://math.univ-angers.fr/perso/lmenichi/String_Classifiant09.pdf. MR 2980450

[CV06] Ralph L. Cohen and Alexander A. Voronov, *Notes on string topology*, String topology and cyclic homology (Basel), Adv. Courses Math. CRM Barcelona, Birkhäuser, 2006, available at http://arxiv.org/pdf/math/0503625v1, pp. 1–95. MR 2240287

[God07] V. Godin, *Higher string topology operations*, preprint (2007), available at http://arxiv.org/pdf/0711.4859v2.

[Kup11] A.P.M. Kupers, *Constructing higher string operations using radial slit configurations*, in preparation (2011), available at http://math.stanford.edu/~kupers/bodigheimeroperationsv12.pdf.

[Kup13] ———, *Notes from the Awesome Joint Berkeley-Stanford String Topology Seminar*, 2013, available at http://math.stanford.edu/~kupers/lectures.pdf.

[MS06] J.P. May and J. Sigurdsson, *Parametrized homotopy theory*, Mathematical surveys and monographs, vol. 132, American Mathematical Society, 2006.

[Tam09] Hirotaka Tamanoi, *Stable string operations are trivial*, Int. Math. Res. Not. IMRN (2009), no. 24, 4642–4685, available at http://arxiv.org/pdf/0809.4561v1. MR 2564371 (2010k:55015)