Copenhagen Workshop Lectures
April 2013
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1) Morse theory, critical levels, level homology. Poincaré duality.

2) The free loop space ΛM

3) The finite dimensional approximation of Morse

4) Examples and computations. Level homology on ΛS^3

5) Products part 1. Pontrjagin product.
Chas Sullivan product.

6) The search for closed geodesics

7) Products part 2. Cohomology products.

8) Support and critical levels.

9) Index growth and level nilpotence. The nonnilpotent case.

10) Products when all geodesics are closed.

11) Computations for spheres

Nancy Hingston

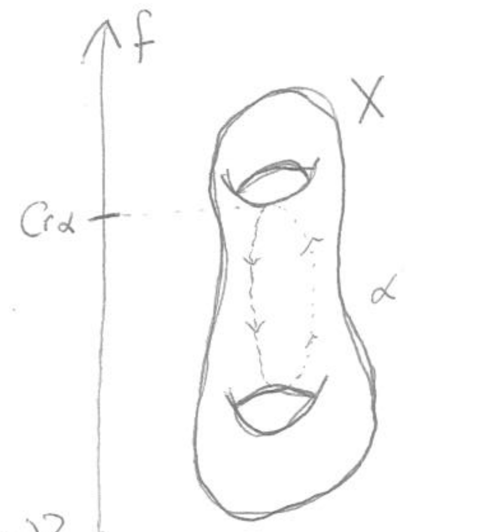
Copenhagen Workshop

Lectures

joint work with

Mark Goresky

Morse theory, Critical Levels
Level homology, Poincaré duality



X compact manifold $f: X \to \mathbb{R}$

$[\alpha] \in H_i(X)$ (so α is a cycle on X)

$$Cr[\alpha] = \inf\{a \in \mathbb{R} \mid [\alpha] \in \text{Image } H_i(X^{\le a}) \to H_i(X)\}$$

$$X^{\le a} \equiv f^{-1}(-\infty, a]$$

Critical level of α

Original definition (Birkhoff) (What a great idea!)
(more or less)

$$Cr[\alpha] = \inf_{[\beta]=[\alpha]} \sup_{x \in \operatorname{Im}\beta} f(\beta). \quad \text{(Thus Mini-Max!)}$$

Thm (Birkhoff) $Cr[\alpha]$ is a critical point of f

Morse theory: Homology of $X \leftrightarrow$ Critical points of f

$$H_k(X) \leftrightarrow \text{Critical points of index } k$$

the level homology $\check{H}_*(X^{\le a}, X^{<a}) = \lim\limits_{\varepsilon \to 0^+} H_*(X^{<a+\varepsilon}, X^{<a})$

Čech homology — singular homology

is the basic building block of Morse theory.

If a is a regular value of f, then the level homology is

$$\check{H}_*(X^{\leq a}, X^{<a}) = H_*(X^{\leq a}, X^{<a}) = 0.$$

If $X^{=a}$ contains exactly one nondegenerate critical point of index k, and no other critical points,

define

$$\check{H}_j(X^{\leq a}, X^{<a}) = H_j(X^{\leq a}, X^{<a}) = \begin{cases} \mathbb{Z} & j=k \\ 0 & \text{otherwise} \end{cases}$$

In both these cases $X^{\leq a}$ has the homotopy type of a finite simplicial complex. But if a is a degenerate critical point ~~[illegible]~~ ~~not~~ even (I think) an isolated degenerate critical point, this might not be true. This is the case where it is necessary to distinguish b/n Čech and singular homology

Try not to emphasize these technical points
Maybe ignore this altogether.

How the building blocks are put together:
Assume critical values of f are discrete.

then f determines a filtration of $C_*(X)$ (singular chains)

There is a SS converging to $H_*(X)$ whose E^1 page is the total level homology

$$\bigoplus_{a\in Cr(f)} \check{H}(X^{\leq a}, X^{<a})$$

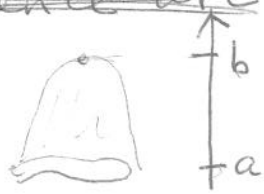
~~Make out this detail!~~

An intermediate filtration gives another SS converging to $H_*(X)$ whose E^1 term is a sum of things of the form

$$H_*(X^{\leq b}, X^{\leq a}) \qquad a, b \text{ regular values}$$

~~The differentials in the spectral sequence are~~

represented by "Morse chains"



Chains supported in $X^{\leq b}$, with boundaries in $X^{\leq a}$. The differentials in the SS come from connecting homomorphisms in the Long Exact S's involving the $H_*(X^{\leq b}, X^{\leq a})$. On the chain level the "boundary" of a Morse

~~chain~~

N = dim X. The Kronecker product $H^k(X^{\le b}, X^{\le a}) \times H_k(X^{\le b}, X^{\le a}) \to \mathbb{Z}$ is the intersection pairing $H_k(X^{\ge a}, X^{\ge b}) \times H_k(X^{\le b}, X^{\le a}) \to \mathbb{Z}$

chain is its chain-boundary in $C_*(X^{\le a})$.

Poincaré duality

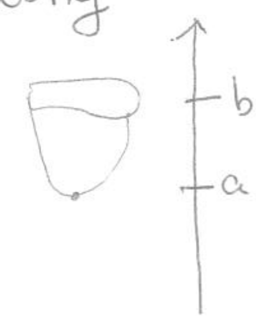
Assume a, b are regular values (or nondeg. crit. values)

then

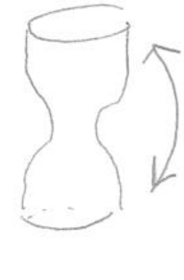
$$\begin{array}{ccc} H^k(X^{[a,b]}, X^{=a}) & \underset{PD}{\cong} & H_{N-k}(X^{[a,b]}, X^{=b}) \\ \text{excisin} \ \| & & \| \ \text{excism} \\ H^k(X^{\le b}, X^{\le a}) & & H_{N-k}(X^{\ge a}, X^{\ge b}) \end{array}$$

this group is generated by chains supported in $X^{\ge a}$, with boundaries in $X^{\ge b}$.

Cohomology is computed using Morse cochains



Poincare duality = Turning X upside-down Using $-f$ instead of f



The Free Loop Space

M compact Riemannian manifold

has a metric

$g: T_xM \times T_xM \to \mathbb{R}$

$(x \in M)$

Probably it is enough to assume a Finsler metric (not necessarily symmetric) These are very interesting for reasons we may get to later. Probably we want M orientable.

$$\Lambda M = \mathrm{Maps}(S^1, M) \qquad S^1 = \mathbb{R}/\mathbb{Z} = SO(2)$$

If we take ΛM to be the H^1-maps (Sobolev-type condition) then ΛM is a complete Hilbert (∞ dim'l) manifold.

The Continuous loops $C^0(S^1, M)$, the H^1 loops $H^1(S^1, M)$ and the C^∞ loops $C^\infty(S^1, M)$ all have the same homotopy type.

Metric g on $M \rightsquigarrow$ Energy $E: \Lambda M \to \mathbb{R}$

$$E(\gamma) = \int |\dot\gamma(t)|^2 dt$$

We will use the function $f = \sqrt{E} : \Lambda M \to \mathbb{R}$

$f(\gamma) \geq \mathrm{length}(\gamma)$ (in the metric g)

$= \mathrm{length}(\gamma) \iff \gamma$ is parameterized proportional to arc length (constant speed) the gradient ∇f is defined. (If $\gamma \in \Lambda M$ $(\equiv H^1(S^1, M))$

then $\nabla f \in T_\gamma \Lambda M = H^1$ vecter fields along γ.)

$$\nabla f(\gamma) = 0 \iff \begin{cases} f = 0 \iff \gamma \text{ is a constant or trivial loop} \\ \text{OR} \\ \gamma \text{ is a closed geodesic, ppal} \end{cases}$$

ΛM is not finite dimensional, but we can do Morse theory in ΛM using the finite dimensional approximation of Morse / Milnor (see next page) or in Hilbert space setting, since f satisfies condition C of Palais-Smale.

Note: If $\gamma \in \Lambda M$ is a critical point of f, then $f(\gamma) = \text{length}(\gamma)$.

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Finite Dimensional approximations   Short version

The finite dimensional approximation of Morse

If points $x, y \epsilon M$ lie at a distance less than the injectivity radius of M, we will write $|x-y|$ for this distance. Given $N \epsilon \mathbb{Z}^+$ and ρ less than half the injectivity radius of M, let $a = \sqrt{N}\rho$ and

$$\mathcal{M}_N^a = \mathcal{M}_N^{\sqrt{N}\rho} = \{(x_0, ..., x_N) : x_i \epsilon M (\forall i); \quad x_0 = x_N \ ; \ \sum_{i=1}^{N} |x_i - x_{i-1}|^2 \leq \rho^2\} \subset M^N.$$

This is the finite dimensional approximation of Morse. There is a natural inclusion map

$$\imath : \mathcal{M}_N^a \hookrightarrow \Lambda \qquad (*)$$

taking $\mathbf{x} = (x_0, ..., x_N)$ to the loop $\imath(\mathbf{x}) = x : S^1 = \mathbb{R}/\mathbb{Z} \to M$ with $x(i/N) = x_i$, and geodesic on each interval $[i/n, (i+1)/N]$. Since the energy of x is

$$E(\mathbf{x}) =: E(x) = N \sum |x_i - x_{i-1}|^2,$$

the image of the map $(*)$ lies in

$$\Lambda^a =: E^{-1}[0, a].$$

Proposition (Morse): The inclusion

$$\mathbf{i} : (\mathcal{M}_N^a, \mathcal{M}_N^0) \to (\Lambda^a, \Lambda^0)$$

is a homotopy equivalence. The critical points of the restriction E to $\mathcal{M}_N^a$ are precisely the closed geodesics of length $\leq a$ on M. The index and nullity (maximal dimension of a subspace of the tangent space at a critical point on which the second derivative is negative definite (resp. $= 0$)) of a closed geodesic as a critical point in $\mathcal{M}_N^a$ are the same as its index and nullity as a critical point in Λ.

Proof of lemma: Λ^a retracts onto $\mathcal{M}_N^a$ leaving $\mathcal{M}_N^a$ fixed: Let $\gamma = \gamma(t)$ be a loop in Λ^a. The restriction of γ to $[i/N, (i+1)/N]$ has energy at most $a^2 = N\rho^2$ and thus length at most ρ. For $s \epsilon [0, 1]$, we replace the restriction of γ to each $[i/N, (i+s)/N]$ by a geodesic with the same endpoints to get the curve $\gamma(s, t)$. Note that $\gamma(s, t) = \gamma(t)$ for all t if $\gamma \epsilon \mathcal{M}_N^a$, and that

$$\begin{aligned} \mathbf{r}(\gamma) &=: \gamma(1, *) \\ &= (\gamma(0), \gamma(1/N), ..., \gamma(1)) \epsilon \mathcal{M}_N^a \end{aligned}$$

for all $\gamma \epsilon \Lambda^a$. The second derivative of the energy is positive definite on the space of vector fields vanishing at x_0, ..., x_N .

Examples/ Computations

1. Quick computation of $\mathbb{Z}$-modules

$H_*(\Omega S^n)$, $H_*(\Lambda S^n)$ Ω = based loops

or maybe

$H_*(\Omega S^3)$, $H_*(\Lambda S^3)$

as in Bott-Tu, Differential Forms in Algebraic Topology, p. 203

2. Morse theoretic computation of level homology of ΩS^n, ΛS^n or maybe ΩS^3, ΛS^3 or maybe ΩM, ΛM where M is a compact, oriented manifold all of whose geodesics are closed with the same least period. [Examples: S^n, $\mathbb{C}P^n$, $\mathbb{H}P^n$ with standard metric. Conjecture of Berger says that if M is simply connected and all geodesics are closed, they must have the same least period.]

In these examples the critical points are degenerate, but form nondegenerate critical manifolds in the sense of Bott. If $A \subset X$ is such a critical submanifold, suitably orientable (i.e. A orientable and the negative vector bundle is orientable) of index λ, then the level homology is

$$H_*(X^{\le a}, X^{<a}) = H_{*-\lambda}(A)$$

In these examples the energy (or f) is a perfect Morse function, which means that all differentials in the SS vanish, and $E^1 = E^\infty$.

The index is given by "the number of conjugate points". A fancier way of saying it is: A closed geodesic is a periodic orbit of the (Hamiltonian) geodesic flow on the unit tangent bundle of M. Associated to each such orbit is a path in the symplectic group $Sp(2(n-1), \mathbb{R})$. The index is the intersection number of this path with the cycle $\{\det(P-I)=0\} \subseteq Sp(2(n-1), \mathbb{R})$.
($n = \dim M$. $P \in Sp(2(n-1), \mathbb{R})$)

Example: S^3

First based loops the constant loop at the base point $*$ has index 0, and contributes $\mathbb{Z}$ in dim 0 to $H_*(\Omega S^3)$
level 0

there is a critical manifold $\approx S^2$ of great circles on S^3 beginning at $*$. (Great circle = intersection of S^3 with a 2 plane thru the origin in $\mathbb{R}^4$) the index is 2, so this manifold contributes (at level 2π) $\mathbb{Z}$ in dim 2, and $\mathbb{Z}$ in dim $2+2=4$. Then there is a critical manifold $\approx S^2$ of twice-traveled great circles on S^3 beginning at $*$. these have length 4π, and index 6, so they contribute $\mathbb{Z}$ in dim 6, and $\mathbb{Z}$ in dim $6+2=8$.

... There is a critical manifold $\approx S^2$ of k-times traveled gread circles on S^3 beginning at $*$. they have length $2\pi k$ (critical level) and index $2(2k-1)$

dim	$H_*(\Omega S^3)$	level
10	$\mathbb{Z}$	6π
9		
8	$\mathbb{Z}$	4π
7		
6	$\mathbb{Z}$	4π
5		
4	$\mathbb{Z}$	2π
3		
2	$\mathbb{Z}$	2π
1		
0	$\mathbb{Z}$	0

dim $\uparrow$ level

$H_*(\Omega S^3)$

Next the free loops on S^3

There is a critical manifold $\approx S^3$ of constant trivial loops on S^3 these have index 0 and contribute $\mathbb{Z}$ in dim 0 and $\mathbb{Z}$ in dim $0+3=3$.

There is a critical manifold $\approx STS^3$ (sphere bundle associated to the tangent bundle of S^3) of great circles on S^3. (There is one great circle for each starting point and initial direction.) The critical level is 2π and the index is 2. Note $STS^3 \approx S^3 \times S^2$

⋮

There is a critical manifold $\approx STS^3$ of k-times traveled great circles on S^3 length $2\pi k$ index $2(2k-1)$

dim	$H_*(\Lambda S^3)$	level
11	$\mathbb{Z}$	4π
10	$\mathbb{Z}$	6π
9	$\mathbb{Z}$	4π
8	$\mathbb{Z}$	4π
7	$\mathbb{Z}$	2π
6	$\mathbb{Z}$	4π
5	$\mathbb{Z}$	2π
4	$\mathbb{Z}$	2π
3	$\mathbb{Z}$	0
2	$\mathbb{Z}$	2π
1		
0	$\mathbb{Z}$	0

Some explicit generators

$A \in C_2(\Omega S^3)$ A = Circles great and small beginning at $* \in S^3$ w/ tangent direction $\vec{V}$.

$U \in C_0(\Omega S^3)$

U = const. loop at $*$

Circle ≡ intersection of S^3 with 2 plane

$B \in C_4(\Omega S^3)$ B = Circles beg. at $* \in S^3$

$C \in C_7(\Lambda S^3)$ C = All circles great & small

$E \in C_3(\Lambda S^3)$ E = All const. loops on S^3.

Products

Pontrjagin product

$$H_j(\Omega M) \times H_k(\Omega M) \to H_{j+k}(\Omega M)$$

"Cycles" $X = \{\alpha\} \subset \Omega$
$Y = \{\beta\} \subset \Omega$

$$[X] \circ_\Omega [Y] = [X \circ Y] = [\{\alpha \circ \beta \mid \alpha \in X, \beta \in Y\}]$$

? Question is there an easy way to compute the Pontriagin ring of a sphere? ???

Important Property:

Fix a metric on M ($\rightsquigarrow f: \Omega \to \mathbb{R}$)

$$Cr[X \cdot Y] \leq Cr[X] + Cr[Y].$$

Ex S^3, ~~with the standard metric~~

$$[u] \cdot [A] = [A]. \qquad [u] = id.$$

$[A] \cdot [A] \neq 0$ In fact $[A]^m \neq 0 \ \forall m \geq 1$

$[A]$ is nonnilpotent.

$H_*(\Omega S^3) = \mathbb{Z}[A]$.

$$Cr[A] = ?$$

With standard metric, $Cr[A] = 2\pi$

$$Cr[A] \cdot [A] = Cr[A] = 2\pi$$

Exercise: $A \cdot A \sim B$

$$Cr[A]^{2m-1} = Cr[A]^{2m} = m \cdot 2\pi$$

$[A]$ is nonnilpotent but is level nilpotent

For some m, $Cr[A]^m < m \cdot Cr[A]$.

$[B]$ is level nonnilpotent.

CS product on $H_*(\Lambda)$ (original def)

Assume $X = \{\alpha\}$, $Y = \{\beta\}$ cycles in Λ

Assume $eX \pitchfork eY$ $\quad e: \Lambda M \to M$ evaluation at 0

$$[X] *_{cs} [Y] = [\{\alpha \cdot \beta \mid \alpha(0) = \beta(0)\}]$$

<u>Check:</u> $H_j(\Lambda) \times H_k(\Lambda) \to H_{j+k-n}(\Lambda)$ $\quad n = \dim M$

Examples on S^3

$[C] *_{cs} [E] = [C]$ $\qquad [E] = \text{unit}$

$[A] *_{cs} [A] = 0$ $\quad$ (in fact $H_*(\Omega) *_{cs} H_*(\Omega) = 0$)

$[C] *_{cs} [U] = [B]$

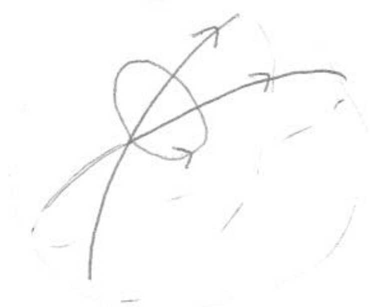
$[C]^{*_{cs} m} \neq 0.$

In the standard metric,

$Cr\ [C]^{*m} = 2m\pi$: level nonnilpotent.

Note $[C]^{*_{CM} m} = [C^{\circ m}]$ (14.5)

where $C^{\circ m} = \{ \gamma_1 \cdot \gamma_2 \cdot \dots \cdot \gamma_m \mid$ each γ_i is a circle and $\gamma_1(0) = \gamma_2(0) = \dots = \gamma_m(0) \}$



$m = 3$

Basic Property of CS product:

For any fixed metric g on M,

$$Cr[X \cdot Y] \leq Cr[X] + Cr[Y]$$

Adult definition (Cohen-Jones, ...) (14.75)

CS product

$$\tilde{\mathcal{F}} = \text{figure-8 space} = \{\gamma \in \Lambda \mid \gamma(0) = \gamma(\tfrac{1}{2})\}$$

$$\Lambda \times \Lambda \xleftarrow[i]{} \tilde{\mathcal{F}} \xrightarrow[j]{} \Lambda \qquad \text{(2 codim } n \text{ embeddings)}$$

$$\tilde{\mathcal{F}} \simeq \{(\alpha,\beta) \in \Lambda \times \Lambda \mid \alpha(0) = \beta(0)\}$$

$\tilde{\mathcal{F}}$ has a tubular nbhd in $\Lambda \times \Lambda$

$$H_*(\Lambda \times \Lambda) \xrightarrow{i^!} H_{*-n}(\tilde{\mathcal{F}}) \xrightarrow{j_*} H_{*-n}(\Lambda)$$

"intersect with $\tilde{\mathcal{F}}$"   inclusion

More precisely

$$H_*(\Lambda \times \Lambda) \to H_*(\Lambda \times \Lambda, \Lambda \times \Lambda - \tilde{\mathcal{F}}) \xrightarrow[\text{Thom}]{} H_{*-n}(\tilde{\mathcal{F}}) \xrightarrow[j_*]{} H_{*-n}(\Lambda)$$

Wheel the patient into the operating room, cover up everything except $\tilde{\mathcal{F}}$

Intersect with $\tilde{\mathcal{F}}$

Note the idea is the same as the original definition of Chas-Sullivan: Given cycles $A \in C_j(\Lambda)$ and $B \in C_k(\Lambda)$, and "assuming transversality",

$$A \times B \in C_{j+k}(\Lambda \times \Lambda)$$

"intersect with $\widetilde{F}$", that is,

$$(A \times B) \cap \widetilde{F} = \{(\alpha, \beta) \mid \alpha \in A, \beta \in B \text{ and } \alpha(0) = \beta(0)\}.$$

the map j_* ~~does~~ takes (α, β) to $\alpha \cdot \beta \in C_{j+k-n}(\Lambda)$.

The transversality condition is very difficult to make rigorous! But the definition of Cohen-Jones is rigorously defined on the level of homology.

"Adult version" of the CS product

due to Cohen, Jones, ...

Difficulties with original definition.

$H_*(\Lambda)$ finitely generated CS ring for spheres, projective spaces; computed by Cohen-Jones-Yan. Describe their method in general terms.

Maybe describe explicitly the CS ring when M is a sphere (or S^3) in terms of the generators above.

Maybe describe the explicit generators for projective spaces given in Hingston, "Loop products on Connected Sums of Projective spaces", in A Celebration of the Mathematical Legacy of Raoul Bott, A.M.S. (page 164-165) there is a geometric reason why the classes have representatives of this form, related to Bott - Samelson ideas. It was known that one could get representative for $H_*(\Lambda S^n)$ using "broken geodesics" since Morse. These generators for $H_*(\Lambda S^n)$ are quite clearly (once the CS product is defined!) CS products of a finite number of generators.

Geometry, and in particular the principle of Poincaré Duality in the free loop space, point to the existence of a product on $H^*(\Lambda M)$ of degree $n-1$.

GEOMETRY

<u>Question</u>: What does the CS product "pick up"? What does the geometry look like when CS products are nontrivial?

<u>The seach for Closed Geodesics</u> (Poincaré, Birkhoff, Morse,...)

Given a compact Riemannian (or Finsler) manifold M, (that is, given a compact manifold with a Riem. or Finsler metric), we look for periodic geodesics "closed geodesics".

round

Now on the standard sphere, <u>all</u> geodesics are closed (there is a great circle thru every point in every direction), but if we perturb the metric slightly it might not be obvious that there are any closed geodesics at all. But this is what Morse invented Morse theory for!

$$f = \sqrt{E} : \Lambda M \to \mathbb{R}$$

$$\text{critical points of } f \equiv \text{closed geodesics on } M$$

Morse theory: $H_*(\Lambda M) \longleftrightarrow$ Critical points of f

$$H_k(\Lambda M) \longleftrightarrow \text{Critical points of index } k$$

We talked before about using the known critical points of f for ~~a special~~ the round metric to compute $H_k(\Lambda S^n)$

But $H_*(\Lambda M)$ does not depend on the metric.

Fix a different (not so round) metric g on S^n.

Use $H_*(\Lambda S^n)$ to find a lower bound for the number of closed geodesics on S^n in the metric g.

"Reminder" how the Morse theory correspondence works:

Given a homology class, e.g. $[A] \in H_*(\Lambda S^3)$

Given a metric G on S^3, try to find a representative of $[A]$ using curves that are as short as possible.

When you get stuck and can't find a representative for $[A]$ with curves of length $< L$, this means there is a closed geodesic of length l.

(minimax construction of Birkhoff.)

As we have seen, $H_*(\Lambda S^n)$ has nontrivial homology in infinitely many dimensions. Does this give infinitely many closed geodesics for any metric?

<u>No</u> Problem of <u>iterates</u>: If γ is a closed geodesic, so are $\gamma^2, \gamma^3, \gamma^4, \dots$ where $\gamma^m(t) = \gamma(mt)$. These are different closed geodesics, with different length, different index. One closed geodesic looks like

ΛM

an army of critical points in ΛM.

<u>Q</u> Is there an <u>algebraic</u> operation on $H_*(\Lambda)$ that corresponds to iteration?

CS powers on $H_*(\Lambda)$ $\rightsquigarrow$ iteration of closed geodesics

$[x], [x]\cdot[x], [x]\cdot[x]\cdot[x]$ ..... when index growth is minimal.

Bott formula for the index of iterates (19) | $n = \dim M$

$$m \cdot \mathrm{Index}(\gamma) - (m-1)(n-1) \le \mathrm{Index}(\gamma^m) \le m \cdot \mathrm{Index}(\gamma) + (m-1)(n-1)$$

comes from intersection theory in symplectic group

minimal index growth

maximal index growth

Ex. Ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{u^2}{c^2} + \frac{v^2}{d^2} = 1$ $\quad a < b < c < d$, $a \simeq d \simeq 1$

There are $\binom{4}{2} = 6$ "short" closed geodesics $\leftrightarrow$ intersection of M with standard coord planes

Shortest γ_s has $\mathrm{index}(\gamma_s) = 2$
$\mathrm{index}(\gamma_s^2) = 6$ max growth
$\mathrm{index}(\gamma_s^3) = 10, 14, 18, \dots$

Longest γ_ℓ has $\mathrm{index}(\gamma_\ell) = 6$ min growth.
$\mathrm{index}(\gamma_\ell^2) = 10$
$\mathrm{index}(\gamma_\ell^3) = 14, 18, 22, \dots$

$\rightarrow$ homology in dim $7, 11, 15, 19, \dots$

These homology classes are $[c], [c]^2, [c]^3, \dots$

CS Product models   Iteration of closed geodesics in case of minimal growth

Cohomology
Product models

Iteration of closed geodesics
in case of maximal growth

Cohomology classes in dim 2, 6, 10, 14, 18, ...
are $\omega, \omega^2, \omega^3, \omega^4, \ldots$ $\omega \in H^2(\Lambda S^3, \Lambda^0 S^3)$

$\boxed{\text{Idea}}$ of the cohomology product:

$$H^*(\Lambda^{\le b}, \Lambda^{\le a}) \cong H_{N-*}(\Lambda^{\ge a}, \Lambda^{\ge b}) \qquad (N = \dim \Lambda M)$$

generated by Morse cochains

Supported in $\Lambda^{\ge a}$, with boundaries in $\Lambda^{\ge b}$

Given $\omega, \tau \in H^*(\Lambda^{\le b}, \Lambda^{\le a})$, find the P duals, take the CS product of the 2 Morse cochains, and take the P dual again to get the product in $H^*(\Lambda)$.

Definition of the cohomology product:

Easier to define the related homology coproduct $\vee$

$$[\vee X, y \otimes z] \equiv [X, y \circledast z]$$

$$\vee X \in H_*(\Lambda) \otimes H_*(\Lambda)$$

$[\ ,\ ]$ = Kronecker product

$\vee$ = coproduct

$\circledast$ = cohomology product

$X \in H_* \Lambda$, $y, z \in H^*(\Lambda, \Lambda^0)$.

$$\widetilde{\mathcal{F}}_{[0,1]} = \{ (\gamma, s) \in \Lambda \times [0,1] \mid \gamma(0) = \gamma(s) \}.$$

$$\Lambda \times [0,1] \underset{\substack{\text{codim} \\ n}}{\hookleftarrow} \widetilde{\mathcal{F}}_{[0,1]} \xrightarrow[\text{map}]{\text{cutting}} \Lambda \times \Lambda$$

$$(\Lambda \times [0,1], \partial(\Lambda \times [0,1])) \longleftarrow (\widetilde{\mathcal{F}}_{[0,1]}, \partial \mathcal{F}_{[0,1]}) \rightarrow (\Lambda \times \Lambda, \partial(\Lambda \times \Lambda))$$

Given $A \in H_*(\Lambda)$

$$A \rightsquigarrow A \times I \in H_*(\Lambda \times [0,1], \partial(\Lambda \times [0,1])$$

$$\underset{\cap \widetilde{\mathcal{F}}}{\rightsquigarrow} H_*(\mathcal{F}_{[0,1]}, \partial \mathcal{F}_{[0,1]}) \xrightarrow[\text{cut}]{} H_*(\Lambda \times \Lambda, \partial(\Lambda \times \Lambda))$$

Given a cycle A, look for all self intersections of loops in A of the form $\gamma(0) = \gamma(s)$. Then apply the cutting map.

Turns out to be a coproduct defined

Cohomology Product:

$$H^i(\Lambda,\Lambda^0)\otimes H^j(\Lambda,\Lambda^0)\to H^{i+j+n-1}(\Lambda,\Lambda^0) \quad \circledast$$

$Cr(x \circledast y) \geq Cr(x)+Cr(y)$ cohomology

$Cr(X \circ Y) \leq Cr(X)+Cr(Y)$ homology

Note reverse inequality!!

Level nilpotence: $x\in H^j(\Lambda,\Lambda^0)$ is <u>level-nilpotent</u>

If for some $m>1$, $Cr(x^{\circledast m}) > m\,Cr(x)$.

Rephrased theorems from differential geometry, illustrating <u>Poincaré duality</u>.

(1) M compact, oriented, dim n. Suppose M carries a metric for which all closed geodesics are <u>"nondegenerate"</u>.

then every homology class in $H_*(\Lambda M)$ is level nilpotent, and every cohomology class in $H^*(\Lambda M,\Lambda^0 M)$ is level nilpotent. (Bott)

$\circledast$ Also $H^i(\Omega,*)\otimes H^j(\Omega,*)\longrightarrow H^{i+j+n-1}(\Omega,*)$

In the nondegenerate case

(generic)

~~Nontrivial~~ powers in local level homology

CS powers: $(H_j(\Lambda^{\le a}, \Lambda^{<a}))^{\otimes m} \longrightarrow H_{mj+(m-1)n}(\Lambda^{\le ma}, \Lambda^{<ma})$

$$X \longmapsto \underbrace{X \cdot X \cdot X \cdot \cdots \cdot X}_{m}$$

Nontrivial $\Leftrightarrow$ minimal index growth (up to a certain level)

AND

cohomology product powers

$H^j(\Lambda^{\le a}, \Lambda^{<a})^{\otimes m} \longrightarrow H^{mj+(m-1)(n-1)}(\Lambda^{\le ma}, \Lambda^{<ma})$

nontrivial $\Leftrightarrow$ maximal index growth (up to a certain level)

(But note: in the nondegenerate case eventually (if m is large enough) these powers will be trivial.)

Thus CS product models iteration when index growth is minimal.

Cohomology product models iteration when index the nonnilpotent case (H.)

Let γ be an isolated closed geodesic of length L.

(i) Assume γ has nonnilpotent level <u>homology</u>.
then for any $\varepsilon > 0$, if $m \in \mathbb{Z}$ is sufficiently large, there is a closed geodesic with length in $(mL, mL+\varepsilon)$.
It follows that M has infinitely many closed geodesics.

(ii) Assume γ has nonnilpotent level <u>cohomology</u>.
Then for any $\varepsilon > 0$, if $m \in \mathbb{Z}$ is sufficiently large, there is a closed geodesic with length in $(mL-\varepsilon, mL)$.
It follows that M has infinitely many closed geodesics.

Products When All Geodesics are Closed.

Suppose that M carries a metric with all geodesics closed and of the same least period. (For example, M is a sphere or projective space with standard metric.)

then the Associated Graded Ring of the homology and cohomology rings can be computed. The energy is a perfect Morse function. For each $m \in \mathbb{Z}^+$, the space of closed geodesics of length mL is a nondegenerate critical manifold $\approx SM$, the unit tangent bundle of M. (For every point $x \in M$, and unit tangent vector $V \in T_x M$, there is a closed geodesic of length mL starting at x with initial tangent vector V.)

The rings are finitely generated, and contain nonnilpotent elements, and thus the products are highly <u>nontrivial</u>.

The complete ring structure can be computed when M is a sphere.