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JENNY WILSON — FI-MODULES AND THE CLASSICAL WEYL GROUPS

GOAL: GENERALIZING FI-Mod TO WEYL GROUPS

WEYL GROUPS

TYPE B/C $\quad B_n \cong$ SIGNED PERMUTATION MATRICES

$B_n \subseteq S_{2n}$, $\quad B_n =$ STABILIZER$(\{-1,+1\}, \dots, \{+n,-n\})$

TYPE D $\quad D_n \hookrightarrow B_n \longrightarrow \mathbb{Z}/2\mathbb{Z}$

$A \longmapsto \#(-1)$'s MOD 2

$D_n =$ EVEN SIGNED PERMUTATIONS

CAT FI_D & FI_{BC}

OBJECT $= \mathbb{N} = \mathbb{Z}_{\geq 0}$

Mor GENERATED BY $\quad \left| \begin{array}{l} \mathrm{End}(\underline{n}) \cong W_n \\ \underline{n} \hookrightarrow \underline{n+1} \end{array} \right.$

TAKE $\underline{n} = \{\pm 1, \dots, \pm n\} \circlearrowleft W_n =$ (EVEN) SIGNED PERMUTATIONS

FI_W-MODULE OVER R IS $V: FI_W \longrightarrow R\text{-Mod}$.

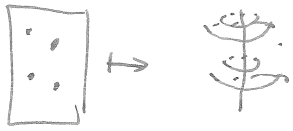
EXAMPLE: (PURE) STRING MOTION GROUP

DEF: Σ_n GROUP OF MOTIONS OF CIRCLES γ_i's IN $\mathbb{R}^3$

GENERATORS:

- α_{ij}: PULL γ_i THROUGH γ_j
- τ_i: TRANSPOSITION i, $i+1$
- σ_i: FLIP i

__Def__: $P\Sigma_n = \langle \alpha_{ij} \rangle$



$$1 \to P\Sigma_n \to \Sigma_n \to \underset{\langle \sigma_i, \tau_i \rangle}{B_n} \to 1$$

$B_n \curvearrowright P\Sigma_n$

__FACT__ $P\Sigma_n$ ARE NOT H_*-STABLE
[JENSEN-McCAMMOND-MEIER '06] $H^m(P\Sigma_n) = \mathbb{Z}^{\binom{n-1}{m} n^m}$

__THM__ Fix m. $R = \mathbb{Z}, \mathbb{Q}$. Then $H^m(P\Sigma_n, R)$ is a FINITELY GENERATED FI_{BC}-MODULE.

__REPRESENTATION THEORY OF B_n / D_n__ (OVER $\mathbb{Q}$)

- B_n IRRED. REPS ARE CLASSIFIED BY DOUBLE PARTITIONS OF n: (λ, μ), $|\lambda| + |\mu| = n$
- B_n CONJUGACY CLASSES ARE CLASSIFIED BY SIGNED CYCLE TYPE.

__EX__: $(-1\ 2\ 1\ -2)$ NEGATIVE 2-CYCLE $\qquad (-1\ 2)(1\ -2)$ POS. 2-CYCLE

__MAIN THM (W)__ LET V BE A f.g. FI_{BC}-MODULE OVER $\mathbb{K}$

- $\mathbb{K}$ NOETHERIAN
  (NOETHERIAN) SUB-FI_W-MODULES ARE f.g.
- $\mathbb{K}$ FIELD
  (POLYNOMIAL DIM) $\exists\, P(T) \in \mathbb{Q}[T]$, $\dim_{\mathbb{K}} V_n = P(n)$ $\forall n \gg 0$
- $\mathbb{K}$ char 0 (REPRESENTATION STABILITY) [CHURCH-FARB]
  — DECOMPOSITIONS OF V_n INTO REPRESENTATIONS STABILIZE FOR $n \gg 0$.

(CHARACTER POLYNOMIAL) $\exists\, P_V \in \mathbb{Q}[X_1, Y_1, \ldots, X_r, Y_r, \ldots]$ ②

~~Sta.~~ $X_r(\sigma) = \#$ POS. r-CYCLES

$Y_r(\sigma) = \#$ NEG r-CYCLES

S.t. $P_V \equiv \chi_{V_n}$ FOR $n \gg 0$.

SAM NARIMAN H_*-STABILITY OF $\mathrm{Diff}(\#_g S^n \times S^n, \mathrm{rel}\ D^{2n})$
MADE DISCRETE

CASE OF SURFACES: $\Sigma_{g,1}$ = SURFACE OF GENUS g WITH 1 BDRY COMPONENT.

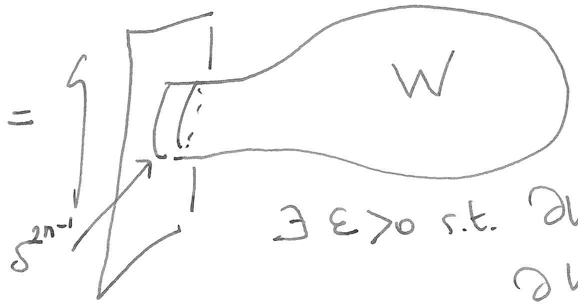
$$\Gamma_{g,1} = \pi_0(\mathrm{Diff}(\Sigma_{g,1}))$$

THM (HARER + ...) $\Gamma_{g,1} \to \Gamma_{g+1,1}$ INDUCES AN ISO ON H_* FOR $* \leq \frac{2g-2}{3}$

$$B\mathrm{Diff}(\Sigma_{g,1}) \xrightarrow{\simeq} B\Gamma_{g,1}$$

HIGHER DIMENSIONS

$$\mathcal{M}_g^{2n}(\mathbb{R}^N) = \Big\{ \quad \Big| \ W \underset{\text{DIFFEO}}{\cong} \overbrace{\#_g S^n \times S^n - \mathring{D}^{2n}}^{W_{g,1}} \Big\}$$



S^{2n-1}

$\exists\, \varepsilon > 0$ s.t. $\partial W \times [0,\varepsilon) \subset W$

$\partial W = S^{2n-1}$

THM: (GALATIUS-R-WILLIAMS, BERGLUND, MADSEN) $n > 2$

$$H_*(\mathcal{M}_g^{2n}) \longrightarrow H_*(\mathcal{M}_{g+1}^{2n}) \text{ IS ISO } * \leq \frac{g-4}{2}$$

QUESTION (MORITA): IS $H_*(B\mathrm{Diff}^\delta(\Sigma_{g,1})) \xrightarrow{\varphi} H_*(B\mathrm{Diff}^\delta(\Sigma_{g+1,1}))$ ISO IN A RANGE?

CLASSIFIES FLAT $\Sigma_{g,1}$-BDLS

BOWDEN: φ IS AN ISO WHEN $* \leq \varepsilon$, $g \geq 8$.

QUESTION (SØREN): SAME QUESTION FOR $B\mathrm{Diff}^\delta(W_{g,1}, \partial)$

REM (THURSTON–HAEFLIGER) $\exists$ SS ~~[illegible]~~

$$E^2_{p,q} \underset{* \leq 3}{\Longrightarrow} H_*(\overline{B}\mathrm{Diff}(M))$$

WHERE $\overline{B}\mathrm{Diff}(M) \longrightarrow B\mathrm{Diff}^\delta(M) \longrightarrow B\mathrm{Diff}(M)$

$\uparrow$ HOMOTOPY FIBRE

THUS $H_*(B\mathrm{Diff}^\delta(W_{g,1}, \partial)) = H_*(\overline{B}\mathrm{Diff}_c(\mathbb{R}^{2n}))$ FOR $* \leq 3$, $g > 10$

c COMPACTLY SUPPORTED

IN PARTICULAR, IT IS STABLE... (FOR $* \leq 3$)

$\underline{K_\bullet(W_{g,1})}$:

$K_p(W_{g,1}) = \{ ((t_0, \varphi_0), \dots, (t_p, \varphi_p)) \mid \varphi_i =$ GERM OF EMBEDDING OF THE CORE OF $W_{1,1}$ INTO $W_{g,1}$ SATISFYING SOME CONDITIONS $\}$

END POINTS

PROP: $|K_\bullet(W_{g,1})|$ is $\lfloor \frac{g-5}{2} \rfloor$ – CONNECTED

$\mathrm{Diff}^\delta(W_{g,1}, \partial) \curvearrowright K_p(W_{g,1})$ NOT QUITE ~~[illegible]~~ TRANSITIVE ~~[illegible]~~.

KRECK CANCELLATION IMPLIES TRANSITIVITY ON VERTICES OR SIMPLICES ENDING AT THE SAME POINT.

$$K_p(W) = \coprod_{\sigma \in I} \frac{\mathrm{Diff}(W_{g,1})}{\mathrm{Diff}(W_{g-p-1,1})_\sigma} \qquad I = \mathrm{Emb}(p+1 \text{ pts}, \mathbb{R}^{2n-1})$$

Def: $X_\bullet(W_{g,1}) := \dfrac{E\mathrm{Diff}^\delta(W_{g,1}) \times K_\bullet(W_{g,1})}{\mathrm{Diff}^\delta(W_{g,1})} = \coprod_{\sigma \in I} B\mathrm{Diff}(W_{g-p-1,1})_\sigma$

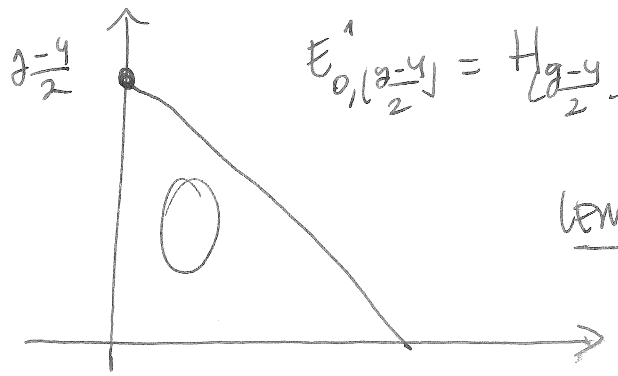
$X_\bullet$ is $\lfloor \frac{g-3}{2} \rfloor$-RESOLUTION FOR $B\mathrm{Diff}^\delta(W_{g,1})$,

i.e. $\|X_\bullet\| \longrightarrow B\mathrm{Diff}^\delta(W_{g,1})$ is $\lfloor \frac{g-3}{2} \rfloor$-conn.

$$(X_\bullet(W_{g+1,1}), X_\bullet(W_{g,1})) \longrightarrow (B\mathrm{Diff}^\delta(W_{g+1,1}), B\mathrm{Diff}^\delta(W_{g,1}))$$

$$E^1_{p,q} = H_q(X_p(g+1), X_p(g)) \Longrightarrow H_{p+q}(\mathrm{Diff}^\delta(W_{g+1,1}), \mathrm{Diff}^\delta(W_{g,1}))$$

AS LONG AS $p+q \le \frac{g-4}{2}$



$$E^1_{0,\lfloor\frac{g-4}{2}\rfloor} = H_{\lfloor\frac{g-4}{2}\rfloor}\Big(\coprod_{\sigma \in I} B\mathrm{Diff}^\delta(W_{g,1})_\sigma, \coprod_{\sigma \in I} B\mathrm{Diff}^\delta(W_{g+1,1})_\sigma\Big)$$

LEMMA:

$$\begin{tikzcd}
\mathrm{Diff}^\delta(W_{g-1,1})_\sigma \arrow[r] \arrow[d] & \mathrm{Diff}^\delta(W_{g,1})_\sigma \\
\mathrm{Diff}^\delta(W_{g,1}) \arrow[r] & \mathrm{Diff}^\delta(W_{g+1,1})
\end{tikzcd}$$

This induces zero on relative H_* when $* \le \frac{g-4}{2}$.

COMPARE:

$$\begin{tikzcd}
\mathrm{Stab}(\{1, g+1\}) \arrow[r] \arrow[d] & \mathrm{Stab}(\{1\}) \arrow[d] \arrow[dl, dashed, "{(1,g+1)}"'] \\
\mathrm{Stab}(\{g+1\}) \arrow[r] & \mathfrak{S}_{g+1}
\end{tikzcd}$$

$\mathfrak{S}_{g+1}$ = sym. group

Thm (N): $H_*(\mathrm{Diff}^\delta(W_{g,1}, \partial)) \longrightarrow H_*(\mathrm{Diff}^\delta(W_{g+1,1}))$ is an iso $* \le \frac{g-4}{2}$