HOMOLOGICAL STABILITY FOR HURWITZ SPACES

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CONFIGURATION - MAPPING SPACES

M ~~TOPOLOGICAL~~ SMOOTH MFD^d, $* \in \partial M$ (if $\partial M \neq \emptyset$)

X TOP. SPACE, $* \in X$

$\mathrm{Conf}_k(M) = \{\underline{m} = (m_1, \dots, m_k) \mid m_i \neq m_j \ \ i \neq j\}$

$S_k \circlearrowright$

Def: $CMap_k(M;X) := \{(\underline{m}, f) \mid \underline{m} \in \mathrm{Conf}_k(M)/S_k,\ f : M \setminus \underline{m} \to X\}$

(M CONNECTED) + PICK $\underline{z} \in \mathrm{Conf}_k \mathring{M}/S_k$

$$\mathrm{Diff}(M) \times_{\mathrm{Diff}(M, \underline{z})} \mathrm{Map}(M \setminus \underline{z}, X) \xrightarrow{\cong} CMap_k(M;X)$$

$$(\varphi, f) \longmapsto (\varphi(\underline{z}), f \circ \varphi^{-1})$$

DEFINES THE TOPOLOGY.

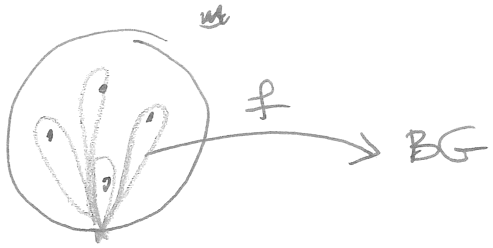
~~[illegible]~~

VARIATIONS: 1) $CMap_k^*$: $f(*) = *$ (PICK $* \in \partial M$ if $\partial M \neq \emptyset$)

2) IF $c \subseteq [S^{d-1}, X]$, WRITE $CMap_k^c(M;X)$ FOR THE SUBSPACE OF $(\underline{m}, f)$ s.t. $[f|_{S^{d-1} \text{ AROUND } m_i}] \in c \quad \forall m_i$.

EX: 1) $M = D^2$, $X = BG$ = CLASSIFYING SPACE OF A FINITE GROUP G.

$$[Y, BG] \cong \left[\begin{matrix} G \to \tilde{Y} \\ \downarrow \text{ PRINC} \\ Y \end{matrix} \right]$$



$$CMap^*_k(D^2, BG) = Diff(D^2,\partial) \times_{Diff(D^2 \setminus \underline{z}, \partial)} Map_*(D^2 \setminus \underline{z}, BG)$$

$$\simeq E\beta_k \times_{\beta_k} (\Omega BG)^{\times k}$$

$$= E\beta_k \times_{\beta_k} G^k$$

β_k GENERATED BY

$\sigma_i = |\,|\ldots \times \ldots|\ldots|$

AND $\sigma_i(g_1, \ldots, g_k) = (g_1, \ldots, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, \ldots)$

$$CMap_k(D^2, BG) = \{(\underline{m}, f)\} \simeq \{(\underline{m}, [f])\} = \{(\underline{m}, \begin{smallmatrix} G \to \tilde{Y} \\ \downarrow \\ Y \end{smallmatrix})\}$$

$$\underset{G \text{ FINITE}}{=} \{(\underline{m}, \begin{smallmatrix} G \to \overline{Y} \\ \downarrow \\ D^2 \end{smallmatrix} \ \text{ISO CLASS OF BRANCHED COVER BRANCHED AT } \underline{m})\}$$

A "MODULI SPACE" OF BRANCHED COVERS OF D^2

EX 2: $M = D^2$, $X = M_g$. $[Y, M_g] \leftrightarrow \left[\begin{smallmatrix} \Sigma_g \to Z \\ \downarrow \\ Y \end{smallmatrix} \right]$

$$CMap_k(D^2, M_g) = \{(\underline{m}, \begin{smallmatrix} \Sigma_g \to Z \\ \downarrow \\ D^2 \setminus \underline{m} \end{smallmatrix})\}$$

IF WE RESTRICT TO $c \in \pi_1 M_g = \Gamma_g = MCG$ THE SUBGROUP OF DEHN TWISTS AROUND A CHOSEN NON-SEPARATING CURVE, ONE CAN "FILL IN" THE BDL AT THE MISSING POINTS BY DEG SURFACES

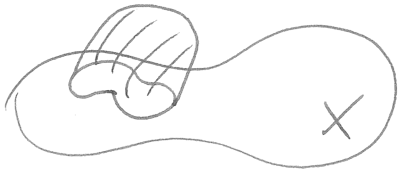
$\rightsquigarrow$ A SPACE OF LEFSCHETZ FIBRATIONS WITH k SINGULAR FIBERS WITH SINGULARITY A PINCHED CURVE.

DEF: LET $A_d(X,c)$ BE THE PUSHOUT OF

$$D^d \times Map^c(S^{d-1},X) \hookleftarrow S^{d-1} \times Map^c(S^{d-1}, X) \longrightarrow X$$

$$(s,f) \longmapsto f(s)$$

THE PUSHOUT FORMALLY ADDS DISCS ALONG MAPS $S^{d-1} \to X$



FOR $d=2$, $X = BG$, WRITE $A_2(BG,c) =: A(G,c)$

$c \in [S^1, BG]$ = CONJUGACY CLASSES in $\pi_1 BG = G$

THM (EVW) $CMap^{c,*}(D^d, X) = \coprod_k CMap_k^{c,*}(D^d,X)$, $d \geq 2$, IS AN E_{d-1} - SPACE, AND

$$\Omega B(CMap^{c,*}(D,X)) \simeq Map_*((D^d, S^{d-1}), (A_d(X,c), X))$$

IDEA OF PROOF: ADAPT EXISTING WORK OF MAY/ MCDUFF/ SALVATORE.

COR: $H_*(Hur_G^c)[\pi_0^{-1}] \overset{\text{GROUP COMPLETION THM}}{\overset{!}{\cong}} H_*(\Omega B(CMap^{c,*}(D^2, BG))) = H_* Map_*((D^2,S^1),(A(G,c),BG))$

$$Hur_G^c = \coprod Hur^c_{G,k} = \coprod CMap_k^{*,c}(D^2, BG)$$

∃ FIBRATION

$$Map_*(S^2, \underbrace{A(G,c)}_{\text{RATIONALLY A WEDGE OF } S^2\text{'s}}) \to Map_*((D^2,S^1),(A(G,c),BG)) \longrightarrow Map_*(S^1,BG) \simeq G$$

$\rightsquigarrow$ LONG EXACT SEQUENCE OF π_*

$\rightsquigarrow \quad 0 \to \pi_2 A(G,c) \to \pi_0 \mathrm{Map}_*((D^2, S^1), (A(G,c), BG)) \to G \to 1$

(*)

COMPUTABLE

DEF: A C-MARKED CENTRAL EXTENSION OF G IS

1) $0 \to K \to \tilde{G} \xrightarrow{f} G \to 1$ (CENTRAL)

2) $\tilde{C} \subseteq \tilde{G}$ s.t. f CARRIES $\tilde{C}$ BIJ TO C.

THM: (*) is A UNIVERSAL C-MARKED CENTRAL EXTENSION.

STABLE HOMOLOGY AS GEOMETRIC ANALYTIC NBR THEORY

JORDAN ELLENBERG

HOW DO WE SHOW A SEQUENCE OF SPACES $X_n = K(\Gamma_n, 1)$ HAS STABLE HOMOLOGY?

- FIND A CPX (APPROXIMATELY CONTRACTIBLE) ON WHICH Γ_n ACTS
- SPECTRAL SEQUENCE
- INDUCTION ON n.

IN EVW09, THE IDEA IS TO PILE ALL THESE SPECTRAL SEQUENCES FOR EACH n ON TOP OF ~~EACH~~ EACH OTHER AND CONSIDER $\bigoplus_n H_i(\mathrm{Hur}^n_{G,c})$ WHICH IS A MODULE FOR $\bigoplus_n H_0(\mathrm{Hur}^n_{G,c}) = R$ HAS A COMBINATORIAL DESCRIPTION

AND THE SPECTRAL SEQUENCE IS A S.S. OF R-MODULES

KEY TECHNICAL FACT: UNDER CERTAIN ASSUMPTIONS BOUNDS ON $\deg H_0^R(M)$ AND $\deg H_1^R(M)$ (AS IN TOM'S TALK II) GIVES NICE BOUNDS ON $\deg H_i^R(M)$ $\forall\, i > 1$ JUST AS FOR FI-MODULES.

IN PUTMAN'S PROOF OF STABILITY FOR CONGRUENCE SUBGROUPS, IN SOME SENSE THE SAME ARGUMENT APPLIED BUT THE CATEGORY IS THAT OF FI-MODULES. (CEFN)

WHY ARE NBR THEORISTS INTERESTED?

QUESTION: HOW MANY SOLUTIONS DOES AN EQUATION OVER $\mathbb{F}_q$ HAS?

i.e. HOW MANY $\mathbb{F}_q$-RATIONAL POINTS DOES A VARIETY X HAVE?

eg $\mathbb{P}^n = \{(x_0, \dots, x_n) - 0 / \text{SCALAR MULT}\}$

Conf^n / S_n = THE SPACE OF MONIC SQUARE FREE DEGREE n POLYNOMIALS IN ONE VARIABLE.

$= \{(a_1, \dots, a_n) - V(\Delta)\}$

WHERE $V(\Delta)$ IS THE LOCUS OF $a_1, \dots, a_n$ s.t. THE POLYNOMIAL $x^n + a_1 x^{n-1} + \dots + a_n$ IS NOT SQUAREFREE

i.e. WHERE $\Delta(a_1, \dots, a_n) = 0$

Δ = DISCRIMINANT

(THE ROOTS ARE n DISTINCT UNORDERED NUMBERS)

$$|\mathbb{P}^n(\mathbb{F}_q)| = \frac{q^{n+1}-1}{q-1} \sim \left(1 - \frac{1}{q}\right)^{-1} q^n$$

$$|\mathrm{Conf}^n(\mathbb{F}_q)| = q^n - q^{n-1} = \left(1 - \frac{1}{q}\right) q^n \qquad \forall n > 1$$

THE POINT COUNTS HAS A NICE ASYMPTOTIC

NOTE ALSO:

- THE SEQUENCE OF VARIETIES $\mathbb{P}^n(\mathbb{C})$ HAS STABLE COHOMOLOGY: $H^i(\mathbb{P}^n; \mathbb{Q}) = \begin{cases} \mathbb{Q} & i \text{ EVEN} \\ 0 & i \text{ ODD} \end{cases}$ $i \leq 2n$.

- SIMILARLY, $\underset{K(Br_n, 1)}{H^i(\mathrm{Conf}^n / S_n(\mathbb{C}), \mathbb{Q})} = \begin{cases} \mathbb{Q} & i = 0, 1 \\ 0 & i > 1 \end{cases}$ STABLY.

LET $X \subseteq \mathbb{P}^n$ A VARIETY OVER $\overline{\mathbb{F}}_p$

$F: X \longrightarrow X$ GIVEN BY

$F(x_0 : \dots : x_n) = (x_0^q : \dots : x_n^q)$

(RECALL $(x+y)^q = x^q + y^q$ IN CHAR p)

ALSO, $\mathbb{F}_q = \{\alpha \in \overline{\mathbb{F}}_q \mid \alpha^q = \alpha\} \longrightarrow X(\mathbb{F}_q) = \mathrm{Fix}(F)$

GROTHENDIECK – LEFSCHETZ TRACE FORMULA (+ POINCARÉ DUALITY)

$$|X(\mathbb{F}_q)| = \sum (-1)^i \operatorname{Tr}\left(F \circlearrowright H^i_{et}(X_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)\right) \quad (*)$$

FOR A SEQUENCE OF VARIETIES X_n, SUPPOSE

- (STABILITY CONDITION) $H^i_{et}(X_{n, \overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ IS STABLE FOR $i < \alpha n$
- (NEGLIGIBILITY) H^i_{et} DOES NOT CONTRIBUTE MUCH TO $(*)$ FOR $i > \alpha n$ (~~BECAUSE~~ THEORY OF WEIGHTS)

EG FOR $\mathbb{P}^n$,

	H^0	H^1	H^2	H^3	H^4	...
TRACE OF FROBENIUS	q^n	0	q^{n-1}	0	q^{n-2}	

$\rightsquigarrow 1 + q^{-1} + q^{-2} + \dots$ IN THE LIMIT.