STABLE HOMOLOGY VIA FUNCTOR HOMOLOGY

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LECTURE I

AIM: GENERAL METHOD TO COMPUTE STABLE HOMOLOGY WITH TWISTED COEFFICIENTS

I SURVEY OF THE RESULTS

$(\mathcal{C}, *, 0)$ SMALL SYMMETRIC MONOIDAL CATEGORY s.t. 0 IS THE INITIAL OBJECT OF $\mathcal{C}$

FOR $X \in \mathcal{C}$, $\quad Aut_{\mathcal{C}}(\underbrace{X * X * \cdots * X}_{n})$

WE HAVE A GROUP HOMOMORPHISM

$$\begin{array}{ccc} Aut_{\mathcal{C}}(X^{*n}) & \longrightarrow & Aut_{\mathcal{C}}(X^{*n+1}) \\ \varphi & \longmapsto & \varphi * Id \end{array}$$

LET $F : \mathcal{C} \longrightarrow \mathbb{K}\text{-Mod}$ ($\mathbb{K}$ COMMUTATIVE RING)

$F(X^{*n}) \longrightarrow F(X^{*(n+1)})$ EQUIVARIANT

↳ INDUCED BY $i : X^{*n} \simeq X^{*n} * 0 \xrightarrow{Id * (0 \to X)} X^{*n} * X \simeq X^{*n+1}$

$$\cdots \to H_*(Aut_{\mathcal{C}}(X^{*n}); F(X^{*n})) \longrightarrow H_*(Aut_{\mathcal{C}}(X^{*(n+1)}), F(X^{*(n+1)})) \searrow \cdots$$

QUESTION 1: DOES THIS SEQUENCE STABILIZE?

i.e. $\forall * \in \mathbb{N}$, $\exists N \in \mathbb{N}$ s.t $\forall n \geq N$,

$$H_d(Aut_{\mathcal{C}}(X^{*n}), F(X^{*n})) \simeq H_d(Aut_{\mathcal{C}}(X^{*(n+1)}), F(X^{*(n+1)}))$$

QUESTION 2 : CAN WE COMPUTE THE COLIMIT OF THIS SEQUENCE ? — THIS COLIMIT IS CALLED THE "STABLE HOMOLOGY OF THE FAMILY $G_n = Aut_{\mathcal{C}}(X^{*n})$ WITH COEFF GIVEN BY THE FUNCTOR F" AND IS DENOTED $H_*(G_\infty, F_\infty)$.

THIS LECTURE SERIES IS CONCERNED WITH Q2.

EXAMPLES :

1) $Sets^f$: FINITE SETS

$* :=$ DISJOINT UNION

$0 := \emptyset$, $X = \{*\}$

$Aut(X^{*n}) = S_n$ = SYMM GROUP

2) R A RING

$P(R)$ PROJECTIVE R-MODULES OF FINITE TYPE

$* := \oplus$

$0 := 0$, $X = R$

$Aut(X^{*n}) = Gl_n(R)$

3) $Eq^{deg}(k)$ FINITE DIMENSIONAL QUADRATIC SPACES OVER k

$* := \perp$

$0 := 0$, $X = H :=$ HYPERBOLIC PLANE $= k^2$ WITH QUAD FORM

$$\begin{aligned} k^2 &\longrightarrow k \\ (x,y) &\longmapsto xy \end{aligned}$$

$Aut_{\mathcal{C}}(X^{*n}) = O_{n,n}(k)$

4) gr = FINITELY GENERATED FREE GROUP

$*$: = $*$ = FREE PRODUCT

$0 := 0$

~~X~~ $:= \mathbb{Z}$

$\mathrm{Aut}(X^{*n}) = \mathrm{Aut}(F_n)$ $\quad F_n = \mathbb{Z}^{*n}$ FREE GROUP ON n LETTERS

FOR $G : \mathcal{C}^{op} \longrightarrow \mathbb{K}\text{-mod}$

IF 0 IS THE TERMINAL OBJECT $\leadsto$ SIMILAR SEQUENCE OF HOMOLOGY GROUPS AND CAN ASK QUESTIONS 1 & 2.

FOR $B : \mathcal{C} \times \mathcal{C}^{op} \longrightarrow \mathbb{K}\text{-mod}$

SAME IF 0 IS THE NUL OBJECT.

	F CONSTANT	F COVARIANT $F(0) = 0$	G CONTRAVARIANT $G(0) = 0$	B BIVARIANT $B(0,-) = B(-,0) = 0$
S_n	NAKAOKA '60 NON-TRIVIAL $H_*(-;F)$	BETLEY '02 $F : \Gamma \longrightarrow \mathbb{K}\text{-mod}$ (Γ = FINITE POINTED SETS) $H_*(S_\infty, F_\infty) = \bigoplus_{i \in \mathbb{N}} H_*(S_\infty \times S_i, cr_i F([1]^{\cdot i}))$	? ($\mathcal{C}$ = FINITE POINTED SETS)	?
GL_n	QUILLEN '72 k FINITE FIELD char $k = p$ $H_*(GL_\infty(k), \mathbb{F}_p) = \begin{cases} \mathbb{F}_p \\ 0 \quad * > 0 \end{cases}$ $H_*(\cdots, \mathbb{F}_\ell) = \cdots$	BETLEY '92 R COMM RING $F : P(R) \to \mathbb{K}\text{Mod}$ POLYNOMIAL $H_*(GL_\infty(R), F_\infty) = 0$	$g \mapsto {}^t g^{-1}$ CONSEQUENCE OF BETLEY	BETLEY/SUSLIN '99 k FINITE FIELD $H_*(GL_\infty(k), B_\infty) = \mathrm{Tor}^{P(k)^{op} \times P(k)}(\mathbb{K}[\mathrm{Hom}_{P(k)}(-,-)], B)$ $= HH_*(P(R), B)$ SCORICHENKO '00 R RING $= HH_*(P(R) \times GL_\infty(R), B$

FOR GL_n: F: $P(R) \xrightarrow{Hom(-,R)} P(R)^{op} \xrightarrow{G} \mathbb{K}\text{-mod}$

THE AUTOMORPHISM $GL_n(R) \longrightarrow GL_n(R)$

$$g \longmapsto {}^t g^{-1}$$

INDUCES AN ISOMORPHISM

$$H_*(GL_n(R), G(R^n)) \cong H_*(GL_n(R), F(R^n))$$

$O_{n,n}$	FIEDOROWICZ-PRIDDY '78 k FIELD CHAR p $H_*(O_\infty(k), \mathbb{F}_p) = \begin{cases} \mathbb{F}_p \\ 0 & *>0 \end{cases}$ p ODD ...	DJAMENT-V '10 k FINITE FIELD DJAMENT '11 RING $F: P(R) \to \mathbb{K}\text{-Mod}$ POLYNOMIAL $H_*(O_\infty, F_\infty) = Tor_*^{P(R)}\left(\mathbb{K}[S^2(V^*)], F\right)$ FINITE FIELD CHAR ODD	CONSEQUENCE	CONSEQUENCE

FOR $O_{n,n}$, A NON-DEG QUAD FORM ON V DEFINES AN ISO $V \to V^*$ $\rightsquigarrow$ $\mathcal{E}q \xrightarrow[e]{\simeq} \mathcal{E}q^{op}$

SO DEDUCE THE CONTRAVARIANT CASE FROM THE COVARIANT CASE

$$H_*(O_\infty, G_\infty) = H_*(O_\infty, G_\infty \circ e).$$

WE DEDUCE THE BIFUNCTOR CASE FROM THE COVARIANT CASE:

$$\mathcal{E}q \xrightarrow{\Delta} \mathcal{E}q \times \mathcal{E}q \xrightarrow{e \times Id} \mathcal{E}q^{op} \times \mathcal{E}q \xrightarrow{B} \mathbb{K}\text{-Mod}$$

$$H_*(O_\infty, B_\infty) \simeq H_*(O_\infty, (B \circ (e \times Id) \circ \Delta)_\infty)$$

$Aut(\mathbb{Z}^{*n}) = A_n$	GALATIUS '11 $H_*(A_\infty, \mathbb{Z}) \xleftarrow{\simeq} H_*(S_\infty, \mathbb{Z})$	DJAMENT-V '12 $F: gr \longrightarrow \mathbb{K}\text{-Mod}$ POLYNOMIAL $H_*(A_\infty, F_\infty) = 0$	? PARTIAL RESULT IN D-V '12 $G: Ab^{op} \to \mathbb{K}\text{-Mod}$ POLYNOMIAL $H_1(A_\infty, G_\infty) = G \otimes_{ab} Id$.

IN DJAMENT-V '10, GENERAL METHOD TO COMPUTE $H_*(G_\infty, F_\infty)$ FROM

- $H_*(G_\infty, \mathbb{K})$
- FUNCTOR HOMOLOGY

THAT CAN RECOVER ALL THE RESULTS IN THE RIGHT 3 COLUMNS OF THE TABLE.

TWO STEPS:

1) IF $\mathcal{C}$ SATISFIES "GOOD PROPERTIES", WE PROVE THAT $H_*(G_\infty, F_\infty)$ CAN BE COMPUTED FROM $H_*(G_\infty, \mathbb{K})$ AND $H_*(\mathcal{C}; F)$

PROBLEM 1: NONE OF THE CATEGORIES $\mathcal{C}$ INTRODUCED PREVIOUSLY SATISFY THESE GOOD PROPERTIES... SO NEED ALTERNATIVE CATEGORIES IN EACH CASE

PROBLEM 2: IN MOST OF THE CASE, WE CAN'T COMPUTE $H_*(\mathcal{C}, F)$ DIRECTLY!

2) COMPUTE $H_*(\mathcal{C}, F)$ (MOST DIFFICULT PART)

GENERAL PRINCIPLE: FIND A CATEGORY $\mathcal{D}$ AND A FUNCTOR $\mathcal{C} \xrightarrow{\varphi} \mathcal{D}$ S.T. WE CAN COMPUTE $H_*(\mathcal{D}, F)$ AND S.T. $H_*(\mathcal{C}, F) \simeq H_*(\mathcal{D}, F)$ FOR F POLYNOMIAL.