LECTURE IV (CHRISTINE)

STEP 2: COMPARISON OF FUNCTOR HOMOLOGIES — THE DIFFICULT PART

WE SAW THAT

$$H_*(GL_\infty(R), F_\infty) \simeq H_*(GL_\infty(R) \times S(R), F)$$

$$H_*(O_\infty, F_\infty) \simeq H_*(O_\infty \times \mathcal{E}_q, F)$$

$$H_*(G_\infty, F_\infty) \simeq H_*(G_\infty \times \mathcal{G}, F)$$

$$G_n = \mathrm{Aut}(\mathbb{Z}^{*n})$$

DIFFICULTY: THE FUNCTOR HOMOLOGY IN $S(R)$, $\mathcal{E}_q$, $\mathcal{G}$ ARE INACCESSIBLE BY DIRECT COMPUTATION!

AIM: FIND A CATEGORY $\mathcal{D}$ WHERE FUNCTOR HOMOLOGY IS ACCESSIBLE, RELATED TO $\mathcal{C}$ (= ONE OF THE PREVIOUS CATEGORIES) BY A FUNCTOR $\varphi : \mathcal{C} \to \mathcal{D}$ s.t.

$$H_*(\mathcal{C}, \varphi^* F) \cong H_*(\mathcal{D}, F) \quad \text{FOR } F \text{ POLYNOMIAL.}$$

① GENERAL METHOD TO PROVE THAT $H_*(\mathcal{C}, \varphi^* F) \cong H_*(\mathcal{D}, F)$

- LEFT KAN EXTENSION FOR $\underset{\mathcal{C}}{\otimes}$:

$\varphi : \mathcal{C} \to \mathcal{D}$

$\varphi^* : \mathcal{D}$-~~Modules~~Mod $\longrightarrow \mathcal{C}$-Mod (BY PRECOMPOSITION)

USUAL LEFT KAN EXTENSION IS THE LEFT ADJOINT OF φ^*:

$$\mathrm{Hom}_{\mathcal{D}\text{-Mod}}(\varphi_! L, K) \simeq \mathrm{Hom}_{\mathcal{C}\text{-Mod}}(L, \varphi^* K) \qquad L \in \mathcal{C}\text{-Mod}, \ K \in \mathcal{D}\text{-Mod}$$

WE HAVE A SIMILAR DEFINITION OF LEFT KAN EXTENSION REPLACING Hom BY $\otimes_{\mathcal{C}}$:

$$\varphi_! : \mathrm{Mod}\text{-}\mathcal{C} \longrightarrow \mathrm{Mod}\text{-}\mathcal{D}$$

$$\text{s.t.} \quad \varphi_!(F) \otimes_{\mathcal{D}} K \simeq F \otimes_{\mathcal{C}} \varphi^*(K) \qquad F \in \mathrm{Mod}\text{-}\mathcal{C}$$

WE CAN DEFINE $\varphi_!$ BY $\varphi_!(F)(a) = F \otimes_{\mathcal{C}} \varphi^*(P_a^{\mathcal{D}})$

GROTHENDIECK SPECTRAL SEQUENCE

$$\mathrm{Mod}\text{-}\mathcal{C} \xrightarrow{\varphi_!} \mathrm{Mod}\text{-}\mathcal{D} \xrightarrow{- \otimes_{\mathcal{D}} K} \mathbb{K}\text{-Mod}$$

WE CONSIDER THE GROTHENDIECK S.S. ASSOCIATED TO THIS COMPOSITION :

$$E^2_{p,q} = \mathrm{Tor}_p^{\mathcal{D}}\big((\mathbb{L}_q \varphi_!)(F), K\big) \Rightarrow \mathrm{Tor}^{\mathcal{C}}_{p+q}(F, \varphi^*(K))$$

WHERE THE LEFT DERIVED FUNCTORS $\mathbb{L}_j \varphi_!$ ARE GIVEN BY

$$(\mathbb{L}_* \varphi_!)(F)(d) := \mathrm{Tor}_*^{\mathcal{C}}(F, \varphi^*(P_d^{\mathcal{D}}))$$

LET $L_*^{\varphi}(G) = \ker\big((\mathbb{L}_* \varphi_!)(\varphi^*(G)) \longrightarrow G\big)$

UNIT OF THE ADJUNCTION

<u>PROP</u>: FOR $G : \mathcal{D}^{op} \longrightarrow \mathbb{K}\text{-Mod}$ AND $K : \mathcal{D} \to \mathbb{K}\text{-Mod}$ s.t. $\mathrm{Tor}_p^{\mathcal{D}}(L_q^{\varphi}(G), K) = 0 \ \forall p, q$, WE HAVE

$$\mathrm{Tor}_*^{\mathcal{C}}(\varphi^*(G), \varphi^*(K)) \xrightarrow{\cong} \mathrm{Tor}_*^{\mathcal{D}}(G, K).$$

IN PARTICULAR, FOR $F = \mathbb{K}$ (CONSTANT FUNCTOR), THE S.S. BECOMES

$$E^2_{pq} = \mathrm{Tor}_p^{\mathcal{D}}((\mathbb{L}_q\varphi_!)(\mathbb{K}), K) \Rightarrow H_{p+q}(\mathcal{C}, \varphi^* K)$$

$$(\mathbb{L}_q\varphi_!)(\mathbb{K})(d) = H_q(\mathcal{C}/d, \mathbb{K})$$

WHERE $\mathcal{C}/d$ IS THE CATEGORY WITH OBJECTS $C \in \mathcal{C}$ EQUIPPED WITH A MORPHISM $d \xrightarrow{f} \varphi(C)$ IN $\mathcal{D}$, AND MAPS $(C, f) \to (C', f')$ MAKING THE OBVIOUS DIAGRAM COMMUTE.

$$L_q^{\varphi}(\mathbb{K}) = \tilde{H}_q(\mathcal{C}/d, \mathbb{K})$$

THE PREVIOUS PROPOSITION SAYS: IF $K : \mathcal{D} \to \mathbb{K}\text{-Mod}$ IS S.t. $\mathrm{Tor}_p^{\mathcal{D}}(L_q^{\varphi}(\mathbb{K}), K) = 0$, THEN

$$H_*(\mathcal{C}, \varphi^*(K)) \xrightarrow{\simeq} H_*(\mathcal{D}, K)$$

② EXAMPLES

- ORTHOGONAL GROUPS

$$\mathcal{E}_q(\mathbb{K}) \xhookrightarrow{\varphi} \mathcal{E}_q^{deg}(\mathbb{K}) = \begin{cases} \text{Obj} = \text{FINITE DIM QUAD SPACES} \\ \text{Mor} = \text{INJECTIVE MAPS OF QUAD SPACES} \end{cases}$$

PROP (CANCELLATION RESULT)
FOR $K : \mathcal{E}_q^{deg}(\mathbb{K}) \to \mathbb{K}\text{-Mod}$ POLYNOMIAL, HAVE

$$\mathrm{Tor}^{\mathcal{E}_q^{deg}(\mathbb{K})}(L_q^{\varphi}(\mathbb{K}), K) = 0$$

BY PREVIOUS PROPOSITION, WE OBTAIN

$$H_*(\mathcal{E}_q, \varphi^*(K)) \xrightarrow{\cong} H_*(\mathcal{E}_q^{deg}, K)$$

A QUADRATIC FORM ON V IS A HOMOGENEOUS POLYNOMIAL OF DEGREE 2 ON V, SO IT IS AN ELEMENT OF $S^2(V^*)$

$$\mathcal{E}_q^{deg}(\mathbb{K}) = (M(\mathbb{K}))_{(S^2)^*}$$

$$\text{obj}: (V, \alpha),\ V \in M(\mathbb{K}),\ \alpha \in S^2(V^*)$$

$$\rightsquigarrow H_*(\mathcal{E}_q^{deg}, F) \simeq \mathrm{Tor}_*^{M(\mathbb{K})}(V \longmapsto \mathbb{K}[S^2(V^*)], F)$$

BY A RESULT OF SUSLIN, FOR F POLYNOMIAL,

$$\mathrm{Tor}_*^{M(\mathbb{K})}(A, F) \simeq \mathrm{Tor}_*^{P(\mathbb{K})}(A, F)$$

SO $H_*(\mathcal{E}_q, \varphi^*(K)) \simeq \mathrm{Tor}_*^{P(K)}(V \mapsto \mathbb{K}[S^2(V^*)], F)$ FOR F POLYNOMIAL

$\rightsquigarrow$ <u>THM</u>: FOR $F: P(\mathbb{K}) \to \mathbb{K}\text{-Mod}$ POLYNOMIAL,

$$H_*(O_\infty, F_\infty) \simeq \mathrm{Tor}_*^{P(\mathbb{K})}(V \mapsto \mathbb{K}[S^2(V^*)], F)$$

SKETCH OF PROOF OF THE <u>CANCELATION RESULT FOR $\mathbb{K}$ A FINITE FIELD [D-V]</u>

$L_q^{\varphi}(K)$ TRANSFORMS INCLUSIONS $V \hookrightarrow \underbrace{V \perp H}_{H \text{ NON-DEG}}$ IN AN ISOMORPHISM.

$\rightsquigarrow$ BY THIS REMARK, WE CAN DEFINE $L_q^{\varphi}(K)$ OVER THE

CATEGORY OF FRACTIONS WHERE WE INVERT
$V \hookrightarrow V \perp H$

$\rightsquigarrow$ EQUIVALENCE OF CATEGORIES BETWEEN THIS CATEGORY OF FRACTIONS AND $\mathrm{Span}(M(\mathbb{K}))$ $\left(\begin{array}{ccc} C & \to & B \\ \downarrow & & \\ A & & \end{array} \right)$

$\rightsquigarrow$ BY A PREVIOUS CANCELATION RESULT OBTAINED BY DJAMENT, WE DEDUCE THE RESULT.

NO MORE TRUE FOR A COMMUTATIVE RING! —— BUT THERE IS AN ALTERNATIVE PROOF [DJAMENT]

- AUTOMORPHISMS OF FREE GROUP $\mathrm{Aut}(F_n)$

$$\mathcal{G} \xrightarrow{\varphi} gr$$

$\mathcal{G}$ = FIN. GEN FREE GROUPS
$A \xrightarrow{u} B$ INJECTIVE, $H \subset B$ s.t. $B \simeq u(A) * H$.

PROP (CANCELATION RESULT) FOR $K : gr \to \mathbb{K}\text{-Mod}$ POLYNOMIAL, $\mathrm{Tor}^{gr}(L_{\mathcal{G}}^{\varphi}(\mathbb{K}), K) = 0$

BY THE PREVIOUS PROPOSITION,
$$H_*(\mathcal{G}, \varphi^*(K)) \xrightarrow{\simeq} H_*(gr, K) = K(0)$$
0 IS THE TERMINAL OBJECT

$\rightsquigarrow$ COMBINING STEP 1 & 2,

THM: FOR $\mathbb{F} : gr \to \mathbb{K}\text{-Mod}$ POLYNOMIAL s.t. $\mathbb{F}(0) = 0$,
$$H_*(G_\infty, F_\infty) = 0.$$

$\mathfrak{a} : gr \longrightarrow \mathbb{K}\text{-Mod}$ (ABELIANIZATION)

1- EXPLICIT PROJECTIVE RESOLUTION OF $\mathfrak{a}$ GIVEN BY THE BAR CONSTRUCTION (ALREADY KNOWN TO JIBLADZE – PIRASHVILI)

$$\cdots \to P_3 \to P_2 \to P_1 \to \mathfrak{a} \qquad P_i = P^{gr}_{\mathbb{Z}^{*i}}$$

2- TAKE THE TENSOR PRODUCT OF THIS RESOLUTION WITH P_r ⇝ OBTAIN A COMPLEX st. THE HOMOLOGY OF THIS COMPLEX IS ISO TO

$$\mathrm{Tor}^{gr}_*(X, \mathfrak{a} \otimes P_r) \quad (*)$$

3- VANISHING CRITERIA, i.e. CONDITIONS ON X st. $(*) = 0$

4- $X = L^{\varphi}_{\bullet}(\mathbb{K})$ SATISFIES THESE CONDITIONS

5. IF $\mathrm{Tor}^{gr}_*(X, \mathfrak{a} \otimes P_r) = 0$, THEN $\mathrm{Tor}^{gr}_*(X, F \circ G) = 0$ WHEN F is POLYNOMIAL, $F(0) = 0$.
(THIS IS A CONSEQUENCE OF A GENERAL RESULT ON THE STRUCTURE OF POLYNOMIAL FUNCTORS ON gr).

<u>CONTRAVARIANT CASE</u> :

$$\mathcal{G} \xrightarrow{\varphi} gr^{op}$$

$$\begin{array}{ccc} A & & A \\ (u,H) \downarrow & \longmapsto & \uparrow u(A) \\ B & & \uparrow \\ & & B \end{array}$$

$$H_*(\mathcal{G}, \varphi^*(G)) \xrightarrow[\text{ISO!}]{\text{NOT AN}} H_*(gr^{op}, G) = 0 \quad \text{IF } G(0) = 0.$$