LECTURE III (CHRISTINE VESPA)

STEP 1 : A GENERAL THEOREM

AIM: COMPUTE $H_*(G_\infty, F_\infty)$ FROM

- $H_*(G_\infty, \mathbb{K})$
- $H_*(\mathcal{C}, F)$

① THE NATURAL MORPHISM $H_*(G_\infty, F_\infty) \longrightarrow H_*(G_\infty \times \mathcal{C}, \pi^* F)$

FOR

$$\begin{tikzcd} \pi : G_\infty \times \mathcal{C} \arrow[r] & \mathcal{C} \arrow[d, "F"] \\ & \mathbb{K}\text{-Mod} \end{tikzcd}$$

GENERAL RESULT:

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left=2, "\varphi"] \arrow[r, shift right=2, "\psi"'] \arrow[r, phantom, "\Downarrow\mu"] & \mathcal{D} \end{tikzcd} \qquad F \in \mathcal{D}\text{-Mod}$$

$$\begin{tikzcd} H_*(\mathcal{C}; \varphi^* F) \arrow[rr, "\mu_*"] \arrow[dr, "\varphi^*"'] & \circlearrowleft & H_*(\mathcal{C}, \psi^* F) \arrow[dl, "\psi_*"] \\ & H_*(\mathcal{D}, F) & \end{tikzcd}$$

APPLICATION: $(\mathcal{C}, \oplus, 0)$ SYM MONOIDAL CAT WHERE THE UNIT IS INITIAL

$$G_n = \mathrm{Aut}_{\mathcal{C}}(X^{\oplus n})$$

$$d_n : G_n \longrightarrow G_{n+1}$$

$$f \in \mathrm{Aut}_{\mathcal{C}}(X^{\oplus n}) \longmapsto f \oplus \mathrm{Id}_X \in G_{n+1}$$

$$\varphi_n : G_n \longrightarrow \mathcal{C} \qquad \text{(CONSIDERING } G_n \text{ AS A CAT WITH 1 OBJ.)}$$

$$* \longmapsto X^{\oplus n}$$

$$\begin{tikzcd} G_n \arrow[r, shift left=2, "\varphi_n"] \arrow[r, shift right=2, "\varphi_{n+1} \circ d_n"'] \arrow[r, phantom, "\Downarrow\mu"] \arrow[dr, "d_n"'] & \mathcal{C} \\ & G_{n+1} \arrow[u, "\varphi_{n+1}"'] \end{tikzcd}$$

μ GIVEN BY $\mathcal{C} \xrightarrow{- \oplus *} \mathcal{C}$

$$\begin{tikzcd}
\cdots \arrow[r] & H_*(G_n, F(X^{\oplus n})) \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\circlearrowright"] & H_*(G_{n+1}, F(X^{\oplus n+1})) \arrow[r] \arrow[dl] & \cdots \\
& & H_*(\mathcal{C}, F) & &
\end{tikzcd}$$

SO WE OBTAIN A NATURAL MAP

$$H_*(G_\infty, F_\infty) \longrightarrow H_*(\mathcal{C}, F)$$

<u>QUESTION</u>: IS THIS MAP AN ISOMORPHISM?

<u>NO!</u> $F = \mathbb{K}$

$$H_*(G_\infty, \mathbb{K}) \longrightarrow H_*(\mathcal{C}, \mathbb{K}) = \begin{cases} \mathbb{K} & * = 0 \\ 0 & * > 0 \end{cases}$$

NOT IN GENERAL AN ISO

$\mathcal{C}$ HAS AN INITIAL OBJECT

<u>IDEA</u>: REPLACE $\mathcal{C}$ BY $\mathcal{C} \times G_\infty$

$$G_n \underset{(\varphi_{n+1} \circ d,\ \text{CAN.})}{\overset{(\varphi_n,\ \text{CANONICAL MAP})}{\rightrightarrows}} \mathcal{C} \times G_\infty \quad \rightsquigarrow \text{SAME AS PREVIOUSLY}$$

$$\rightsquigarrow \quad H_*(G_\infty, F_\infty) \longrightarrow H_*(\mathcal{C} \times G_\infty, \pi^* F)$$

NOW THIS MAP IS AN ISOMORPHISM FOR $F = \mathbb{K}$.

2. HYPOTHESIS ON $\mathcal{C}$

(H1) TRANSITIVITY:

- <u>STRONG VERSION</u>: $\forall A, B \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(A,B)$ is A TRANSITIVE $\mathrm{Aut}_{\mathcal{C}}(B)$-SET.

- WEAK VERSION: $\forall A, B \in \mathcal{C}$, $f: A \to B$, THERE EXISTS AN ISOMORPHISM $\alpha: A \oplus B \to A \oplus B$ MAKING THE FOLLOWING DIAGRAM COMMUTATIVE:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[drr] & B \arrow[r] & A \oplus B \arrow[d, "\alpha"] \\ & & A \oplus B \end{tikzcd}$$

(H2) STABILIZERS $A, B \in \mathcal{C}$

$\mathrm{Aut}_{\mathcal{C}}(B) \longrightarrow \mathrm{Aut}(A \oplus B)$ IS AN INJECTION

ITS IMAGE IS THE SUBGROUP OF AUTOMORPHISMS φ MAKING THE FOLLOWING COMMUTE:

$$\begin{tikzcd} & A \oplus B \arrow[d, "\varphi"] \\ A \arrow[ur] \arrow[r] & A \oplus B \end{tikzcd}$$

(H3) GENERATOR

- STRONG VERSION: THERE EXISTS $X \in \mathcal{C}$ s.t. $\forall A \in \mathcal{C}$, $\exists n \in \mathbb{N}$, $A \simeq X^{\oplus n}$

- WEAK VERSION THERE EXISTS $X \in \mathcal{C}$ s.t. $\forall A \in \mathcal{C}$, $\exists n \in \mathbb{N}$, $\exists B \in \mathcal{C}$ s.t. $A \oplus B \simeq X^{\oplus n}$.

3. THE THEOREM

THM (D-V, '10) LET $(\mathcal{C}, \oplus, 0)$ BE A SYM. MONOIDAL CAT, 0 IS THE INITIAL OBJECT, AND SATISFYING H1-3 (WEAK OR STRONG), THEN

$$H_*(G_\infty, F_\infty) \xrightarrow{\simeq} H_*(\underbrace{G_\infty \times \mathcal{C}}, \pi^* F)$$

LIKE A BOREL CONSTRUCTION

BY KÜNNETH FORMULA,

PROP: $\mathbb{K}$ A FIELD, SAME HYPOTHESIS,

$$H_*(G_\infty, F_\infty) \xrightarrow{\simeq} H_*(G_\infty, \mathbb{K}) \times H_*(\mathcal{C}, F).$$

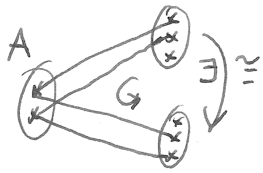
4. THE EXAMPLES

- S_n (SYMM. GROUPS) Set^f DOES NOT SATISFY (H1)

$FI = \Theta = \{$ Obj = FINITE SETS, MORPH = INJECTIONS $\}$ $\quad \oplus = \amalg$, $0 = \emptyset$

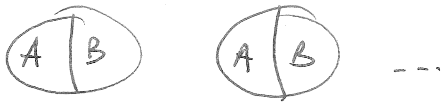
$(\Theta, \amalg, 0)$ is SYM MONOIDAL. $A \in \Theta$, $|A| = n \Rightarrow \mathrm{Aut}_\Theta(A) \cong S_n$

(H1) STRONG: $\mathrm{Hom}_\Theta(A,B)$ IS A TRANSITIVE $\mathrm{Aut}_\Theta(B)$-SET



(BUT NEED INJECTIONS HERE!)

(H2) $\mathrm{Aut}_\Theta(B) \longrightarrow \mathrm{Aut}_\Theta(A \amalg B)$



(H3) STRONG: $X = \{1\} \rightsquigarrow A \cong \{1\}^{\amalg |A|}$

THM $\Longrightarrow$ FOR $F : \Theta \longrightarrow \mathbb{K}\text{-Mod}$,

$$H_*(S_\infty, F_\infty) \xrightarrow{\simeq} H_*(S_\infty \times \Theta, F)$$

~~[illegible]~~ CAN RECOVER BETLEY'S THM FROM HERE.

(IN THIS CASE, ALL FUNCTORS ARE COLIMIT OF POLYNOMIAL FUNCTORS WHICH SIMPLIFIES THINGS).

- LINEAR GROUP $GL_n(R)$

$P(R)$ DOES NOT SATISFY THE HYPOTHESIS
$\llcorner$ f.g PROJ. MODULES.

CONSIDER INSTEAD $S(R) = \{$ PROJ. FIN. GEN. LEFT R-MOD
MORPH : $M \xrightarrow{(u,v)} N$ WHERE $M \xrightarrow{u} N$, $N \xrightarrow{v} M$ s.t. $v \circ u = id_M$

$(S(R), \oplus, 0)$ SYM. MON. CAT.

$M \in S(R)$, $Aut_{S(R)}(M) = \{(u,v) : M \to M\} = GL(M)$.

(H1) (~~STRONG~~ WEAK VERSION) (STRONG VERSION NOT TRUE: IF $R \cong R^2$

$$\begin{tikzcd} R \arrow[r, hook] \arrow[dr, "\cong"'] & R^2 \arrow[d, phantom, "\not\cong"] \\ & R^2 \end{tikzcd}$$

)

$$\begin{tikzcd} M \arrow[r, "(u,v)"] \arrow[drr] & N \arrow[r] & M\oplus N \\ & & M\oplus N \arrow[u, "\begin{pmatrix} 0 & v \\ u & 1 \end{pmatrix}"'] \end{tikzcd}$$

$$\begin{pmatrix} 0 & v \\ u & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & v \\ u & 1-uv \end{pmatrix}$$

<u>REM</u>: THIS IS ALSO TRUE FOR $M(R)$: SAME OBJ, MORPH $M \xrightarrow{u} N$ SPLIT MONOMORPHISMS.

(H2) $Aut_{S(A)}(V) \longrightarrow Aut_A(U \oplus V)$

$$\begin{array}{ccc} Aut_{S(A)}(V) & \longrightarrow & Aut_A(U\oplus V) \\ \| & & \| \\ GL(V) & & GL(U\oplus V) \\ (M, M^{-1}) & \longmapsto & \left(\begin{pmatrix} 1_U & 0 \\ 0 & M \end{pmatrix}, \begin{pmatrix} 1_U & 0 \\ 0 & M^{-1} \end{pmatrix}\right) \end{array}$$

INJECTIVE AND THE IMAGE IS THE GROUP OF AUTO (φ, φ^{-1}) MAKING THE FOLLOWING DIAGRAM COMMUTE:

$$\begin{tikzcd} & U\oplus V \arrow[d, "\left(\begin{pmatrix} 1_U & 0 \\ 0 & M \end{pmatrix}, \begin{pmatrix} I_U & 0 \\ 0 & M^{-1} \end{pmatrix}\right)"] \\ V \arrow[ur] \arrow[r] & U\oplus V \end{tikzcd}$$

Rem: This is no more true in $M(R)$.
In fact, the subgroup of auto φ, (φ split) making the following comm. !

$$\begin{tikzcd} & U \oplus V \arrow[d, "\begin{pmatrix} I_V & A \\ 0 & M \end{pmatrix}"] \\ V \arrow[ur] \arrow[dr] & \\ & U \oplus V \end{tikzcd}$$

(H3) (weak version) $X = R$. $\forall A$ proj. f.g. left R-Mod $\exists n, \exists B, A \oplus B \simeq R^{\oplus n}$.

$\leadsto$ THM $\Rightarrow \forall F : S(R) \to \mathbb{K}\text{-Mod}$,

$$H_*(GL_\infty(R), F_\infty) \xrightarrow{\simeq} H_*(GL_\infty(R) \times S(R), F)$$

But the right hand side is difficult to compute...
$\longrightarrow$ will need a second step.

- Orthogonal groups: k field

$\mathcal{E}_q(k)$ Obj: non-deg quad spaces of fin dim
Morph: linear maps compatible with quad forms.

(H1) (strong) is true by Witt's theorem

Rem: For a ring R, the strong version of H1 is no more true but we have the weak version.

(H2) by Witt's thm or do as ~~before~~ (maps of quadr spaces are canonically split)

(H3) (weak version) H hyperbolic plane, $(x,y) \mapsto xy$
$\forall V \in \mathcal{E}_q$, $\exists W \in \mathcal{E}_q$ s.t. $V \perp W \simeq H^{\perp n}$

$\leadsto$ THM $\Rightarrow \forall F : \mathcal{E}_q(k) \to \mathbb{K}\text{-Mod}$, $H_*(O_\infty(k), F_\infty) \simeq H_*(\mathcal{E}_q(k) \times O_\infty(k); F)$

BUT HERE AGAIN THE RHS is HARD TO COMPUTE.

• AUTOMORPHISMS OF FREE GROUPS $Aut(F_n)$

$\mathcal{G}$: Obj : f.g. FREE GROUPS (gr DOES NOT WORK)

Mor: $A \xrightarrow{(u,H)} B$ s.t. $u : A \to B$ is A GROUP MONOMORPHISM

H is A SUBGROUP OF B s.t. $B \simeq u(A) * H$

Composition :

$$\begin{tikzcd} A \arrow[r, "{(u,H)}"] \arrow[rr, bend right, "{(v\circ u,\ v(H)*K)}"'] & B \arrow[r, "{(v,K)}"] & C \end{tikzcd}$$

$Aut_{\mathcal{C}}(A) = Aut_{Gr}(A)$

H1 STRONG

H2 $Aut_{\mathcal{C}}(B) \longrightarrow Aut_{\mathcal{C}}(A*B)$

$B \xrightarrow{u} B \longmapsto A*B \xrightarrow{Id_A * u} A*B$

$$\begin{tikzcd} & A*B \arrow[d, dashed, "\exists \begin{pmatrix} 1_A & 0 \\ \alpha & \varphi \end{pmatrix}"] \\ A \arrow[ur, "{(u,B)}"] \arrow[r, "{(v,B)}"'] & A*B \end{tikzcd}$$

H3 STRONG. $X = \mathbb{Z}$, $A \simeq \mathbb{Z}^{*n}$

THM $\Rightarrow F : \mathcal{G} \longrightarrow \mathbb{K}\text{-Mod}$

$$H_*(G_\infty, F_\infty) \simeq H_*(\mathcal{G} \times G_\infty, F)$$