LECTURE II (CHRISTINE VESPA)

INGREDIENTS: FUNCTOR HOMOLOGY AND POLYNOMIAL FUNCTORS

1) FUNCTOR HOMOLOGY: (= HOMOLOGY IN FUNCTOR CATEGORIES)

$\mathcal{C}$ SMALL CATEGORY

$\mathbb{K}$ A FIXED COMMUTATIVE RING

Def: • A LEFT $\mathcal{C}$-MODULE IS A FUNCTOR $F: \mathcal{C} \to \mathbb{K}\text{-Mod}$

$\rightsquigarrow$ $\mathcal{C}$-Mod IS THE CATEGORY OF LEFT $\mathcal{C}$-MODULES WITH MORPHISMS THE NATURAL TRANSFORMATIONS.

• A RIGHT $\mathcal{C}$-MODULE IS A FUNCTOR $G: \mathcal{C}^{op} \to \mathbb{K}\text{-Mod}$

$\rightsquigarrow$ Mod-$\mathcal{C}$ = CAT OF RIGHT $\mathcal{C}$-MOD.

Rem: A $\mathcal{C}$-MODULE DETERMINES NOT ONLY A $\mathbb{K}$-MODULE $F(c)$ FOR EACH OBJ c, BUT A $\mathbb{K}[\mathrm{End}_{\mathcal{C}}(c)]$-MODULE (FOR $\mathcal{C} = FI$, $\mathrm{End}_{FI}([n]) = S_n$), AND IT IS MORE THAN A SEQUENCE OF $\mathbb{K}[\mathrm{End}_{\mathcal{C}}(c)]$-MODULES BECAUSE OF PROPERTIES GIVEN BY THE FUNCTORIALITY.

Prop: $\mathcal{C}$-mod AND mod-$\mathcal{C}$ ARE ABELIAN CATEGORIES. (BECAUSE LIMITS AND COLIMITS ARE COMPUTED IN THE TARGET CATEGORY WHICH IS ABELIAN)

STANDARD PROJECTIVE GENERATORS

Def: $\forall c \in \mathcal{C}$, $P_c^{\mathcal{C}} := \mathbb{K}[\mathrm{Hom}_{\mathcal{C}}(c, -)]$

(CORRESPONDS TO $M(d)$ FOR $\mathcal{C} = FI$ IN CHURCH'S TALK)

Ex: IF $\mathcal{C}$ HAS AN INITIAL OBJECT I,

$P_I^{\mathcal{C}} = \mathbb{K}[\mathrm{Hom}_{\mathcal{C}}(I, -)] = \mathbb{K}$ $\rightsquigarrow$ THE CONSTANT

FUNCTOR IS PROJECTIVE
(IN CHURCH'S LECTURE, $M(0) = \mathbb{Z}$ IS THE CONSTANT FUNCTOR)
INITIAL IN FI

PROP:
1) $\mathrm{Hom}_{\mathcal{C}\text{-Mod}}(P_c^{\mathcal{C}}, F) \simeq F(c)$ (BY YONEDA LEMMA)
2) $\forall c \in \mathcal{C}$, $P_c^{\mathcal{C}}$ IS A PROJECTIVE OBJECT IN $\mathcal{C}$-Mod
3) ANY PROJECTIVE OBJECT IN $\mathcal{C}$-Mod IS A DIRECT SUMMAND IN A COPRODUCT OF OBJECTS $P_c^{\mathcal{C}}$
4) FOR ANY OBJECT $F \in \mathcal{C}$-Mod, THERE IS AN EPIMORPHISM $P \to F$ (P IS PROJECTIVE)

$\rightsquigarrow$ THESE GIVE A SET OF PROJECTIVE GENERATORS IN $\mathcal{C}$-Mod. $\{P_c^{\mathcal{C}}\}_{c \in \mathrm{Obj}\,\mathcal{C}}$

SIMILARLY, THE FUNCTORS $P_c^{\mathcal{C}^{op}} = \mathbb{K}[\mathrm{Hom}_{\mathcal{C}}(-,c)]$
$\rightsquigarrow$ PROJECTIVE GENERATORS OF Mod-$\mathcal{C}$.

EX: IF T IS THE TERMINAL OBJECT OF $\mathcal{C}$,
$$P_T^{\mathcal{C}^{op}} = \mathbb{K}[\mathrm{Hom}_{\mathcal{C}}(-,T)] = \mathbb{K} \quad \text{CONSTANT FUNCTOR.}$$

TENSOR PRODUCT

DEF: FOR $F \in \mathcal{C}$-Mod AND $G \in$ Mod-$\mathcal{C}$, WE DEFINE
$$G \otimes_{\mathcal{C}} F := \bigoplus_{c \in \mathrm{Obj}\,\mathcal{C}} G(c) \otimes_{\mathbb{K}} F(c) \Big/ \sim \quad \in \mathbb{K}\text{-Mod}$$
WHERE $\forall f : c \to c'$, $x \in G(c')$, $y \in F(c)$,
$$x \otimes F(f)(y) \sim G(f)(x) \otimes y$$

PROP: $G \otimes_{\mathcal{C}} P_c^{\mathcal{C}} \simeq G(c)$, $P_c^{\mathcal{C}^{op}} \otimes F \simeq F(c)$

PROP: THE BIFUNCTOR $- \otimes_{\mathcal{C}} - : \text{Mod-}\mathcal{C} \times \mathcal{C}\text{-Mod} \to \mathbb{K}\text{-Mod}$ IS RIGHT EXACT IN EACH VARIABLE.
($\Rightarrow$ THE STANDARD PROJECTIVES ARE FLAT)

TOR FUNCTORS

DEF: FOR $F \in \mathcal{C}\text{-Mod}$ AND $G \in \text{Mod-}\mathcal{C}$, WE DEFINE

$$\text{Tor}_i^{\mathcal{C}}(G, F) := H_i(P_\bullet \otimes_{\mathcal{C}} F) = H_i(G \otimes_{\mathcal{C}} P_\bullet)$$

WHERE $P_\bullet = \dashrightarrow P_2 \to P_1$ IS A PROJECTIVE RESOLUTION OF G. (OR P A PROJ. RESOLUTION OF F)

HOMOLOGY OF A CATEGORY

DEF: FOR $F \in \mathcal{C}\text{-Mod}$, WE DEFINE

$$H_*(\mathcal{C}; F) := \text{Tor}_*^{\mathcal{C}}(\mathbb{K}, F).$$

EX: $\mathcal{C} = \left\{ \begin{matrix} * \\ G \end{matrix} \right.$ $\rightsquigarrow$ RECOVERS USUAL GROUP-HOMOLOGY.

EX: • IF $\mathcal{C}$ HAS AN INITIAL OBJECT, I, THEN

$$H_*(\mathcal{C}; \mathbb{K}) = \begin{cases} \mathbb{K} & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

• IF $\mathcal{C}$ HAS A TERMINAL OBJ T, THEN

$$H_*(\mathcal{C}; F) = \begin{cases} F(T) & * = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

QUESTION: IS IT EASIER TO COMPUTE $H_*(\mathcal{C}, F)$

THAN $\mathrm{Colim}_n \left(H_*(\mathrm{Aut}_{\mathcal{C}}(X^{*n}), F(X^{*n})) \right)$.

SOMETIMES YES.

EX: $F : gr \longrightarrow \mathbb{K}\text{-Mod}$ S.T. $F(0) = 0$

$\uparrow$ FINITELY GEN FREE GROUPS

$H_*(gr, F) = 0$ (BECAUSE 0 IS TERMINAL AND $F(0)=0$)

IS $\leftarrow$ F POLYNOMIAL (D-V)

$H_*(A_\infty, F_\infty)$ $\quad A_n = \mathrm{Aut}(\mathbb{Z}^{*n})$

DEF: FOR $B \in (\mathcal{C}^{op} \times \mathcal{C})\text{-Mod}$ A BIFUNCTOR, WE DEFINE THE HOCHSCHILD HOMOLOGY OF $\mathcal{C}$ WITH COEFFICIENTS IN B BY

$$HH_*(\mathcal{C}; B) := \mathrm{Tor}_*^{\mathcal{C}^{op} \times \mathcal{C}}\left(\mathbb{K}[\mathrm{Hom}_{\mathcal{C}^{op}}(-,-)], B \right)$$

WHERE $\mathbb{K}[\mathrm{Hom}_{\mathcal{C}^{op}}(-,-)] : \mathcal{C}^{op} \times \mathcal{C}^{op} \longrightarrow \mathbb{K}\text{-Mod}$

$$(a, b) \longmapsto \mathbb{K}\text{-}[\mathrm{Hom}_{\mathcal{C}}(b, a)]$$

EX: $HH_0(\mathcal{C}, B) = \bigoplus_{c \in \mathrm{Obj}\,\mathcal{C}} B(c,c) / \sim$

EXTERIOR TENSOR PRODUCT

$F \in \mathcal{C}\text{-Mod}$, $\quad G \in \mathcal{D}\text{-Mod}$

$F \boxtimes G \in (\mathcal{C} \times \mathcal{D})\text{-Mod}$, $\quad F \boxtimes G(c,d) := F(c) \otimes G(d)$.

PROP: LET $\mathcal{C}$-Mod $\ni F$ AND $G \in$ Mod-$\mathcal{C}$,

- $HH_0(\mathcal{C}, G \boxtimes F) \simeq G \otimes_{\mathcal{C}} F$
- IF F OR G HAS VALUES IN PROJECTIVE $\Bbbk$-MODULES, THEN $HH_*(\mathcal{C}, G \boxtimes F) \simeq \mathrm{Tor}_*^{\mathcal{C}}(G, F)$.

REM: FOR $F \in \mathcal{C}$-Mod AND $\pi : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$

$$HH_*(\mathcal{C}; \pi^* F) \simeq H_*(\mathcal{C}, F)$$

2 POLYNOMIAL FUNCTORS

$(\mathcal{C}, *, 0)$ SYMM MONOIDAL CATEGORY S.T. THE UNIT 0 IS THE NUL OBJECT.

EX: $(\Gamma, \amalg, \{0\})$, $(P(R), \oplus, 0)$, $(gr, *, 0)$

FINITE POINTED SETS

$F : \mathcal{C} \to \mathcal{A}$, $\mathcal{A}$ AN ABELIAN CATEGORY.

DEF: CROSS-EFFECTS OF F: $cr_n F : \mathcal{C}^{\times n} \longrightarrow \mathcal{A}$

- $cr_1 F(X) = \ker(F(0) : F(X) \longrightarrow F(0))$
- $cr_2 F(X) = \ker\left(F(X_1 * X_2) \xrightarrow[(F(r_1), F(r_2))]{} F(X_1) \oplus F(X_2)\right)$

WHERE $r_1 : X_1 * X_2 \xrightarrow{Id * 0} X_1 * 0 \simeq X_1$

$r_2 : X_1 * X_2 \xrightarrow{0 * Id} 0 * X_2 \simeq X_2$

- $cr_n F(X_1, \dots, X_n) = cr_2(cr_{n-1} F(-, X_3, \dots, X_n))(X_1, X_2)$

FOR $F(0)=0$, HAVE

$$F(X_1 * X_2) = F(X_1) \oplus F(X_2) \oplus cr_2 F(X_1, X_2)$$

$$F(X_1 * X_2 * X_3) = F(X_1) \oplus F(X_2) \oplus F(X_3) \oplus cr_2 \cancel{F}(X_1, X_2) \oplus cr_2 F(X_1, X_3) \oplus cr_2 F(X_2, X_3) \oplus cr_3 F(X_1, X_2, X_3)$$

__PROP:__ $F(0)=0$, $\displaystyle F(X_1 * \dots * X_n) = \bigoplus_{\ell=1}^{n} \bigoplus_{1 \le i_1 < \dots i_\ell \le n} cr_\ell F(X_{i_1}, \dots, X_{i_\ell})$

__DEF__: A FUNCTOR $F: \mathcal{C} \to \mathcal{A}$ IS SAID TO BE __POLYNOMIAL OF DEGREE $\le n$__ IF $cr_{n+1} F = 0$.

__EX__: 1) $F = (-)^{ab} : gr \longrightarrow Ab$

$$F(G*H) = (G*H)^{ab} = G^{ab} \oplus H^{ab} = F(G) \oplus F(H)$$

$\Rightarrow cr_2 F = 0$.

SO F IS POLYNOMIAL OF DEGREE 1.

2) $\mathcal{C} = P(R)$, $M \in P(R)$ FIXED, R COMMUTATIVE

(PROJ. R-MODULES)

$$F = - \otimes M : P(R) \to P(R)$$

$$F(N_1 \oplus N_2) = F(N_1) \oplus F(N_2) \quad \to \text{POLY OF DEGREE 1}$$

$$T^2 : P(R) \longrightarrow P(R)$$

$$M \longmapsto M \otimes M$$

$$T^2(M \oplus N) = T^2(M) \oplus T^2(N) \oplus \underbrace{M \otimes N \oplus N \otimes M}_{cr_2 T^2(M,N)}$$

$cr_2 T$ $\hookrightarrow$ POLYNOMIAL IS POLYNOMIAL OF DEGREE 1 BY PREVIOUS EXAMPLE

T^2 IS POLYNOMIAL OF deg 2 $\Leftarrow$

(AS $cr_3 T^2 = cr_2(cr_2 T^2) = 0$)

$$\left.\begin{array}{l}
\bullet\ T^n(M) = M^{\otimes n} \\
\Gamma^n(M) = (M^{\otimes n})^{S_n} \\
S^n(M) = (M^{\otimes n})_{S_n} \\
\Lambda^n(M) = M^{\otimes n} / \{\dots\otimes m \otimes \dots \otimes m \otimes \dots\}
\end{array}\right\} \text{ ARE POLYNOMIAL OF DEGREE } n.$$

EXTENDED DEFINITION (D-V) OF POLYNOMIAL FUNCTORS $F: \mathcal{C} \to Ab$ WITH $\mathcal{C}$ SYM MON WITH THE UNIT INITIAL OBJECT (NOT NECESSARILY TERMINAL)

— SEE FRIDAY AFTERNOON

EX: $\Theta = FI$, $S(R)$

MORE SYMMETRIC DEFINITION OF $cr_n F$:

$$cr_n F(X_1, \dots, X_n) = \ker\Big(F(X_1 * \dots * X_n) \to \bigoplus_{i=1}^{n} F(X_1 * \dots * \hat{X_i} * \dots * X_n)\Big)$$