ANDY PUTMAN: $H_2(IA_n)$ JOINT WITH M. DAY

$F_n = \langle x_1, \dots, x_n \rangle$

$IA_n = \ker(\mathrm{Aut}(F_n) \longrightarrow GL_n(\mathbb{Z}))$

([NIELSEN: $AI_2 \cong F_2$])

MAGNUS '35: $IA_n$ HAS A FINITE GENERATING SET

$X_n^{IA}$: i) $C_{ij} = \alpha_{ij}$ $(1 \le i, j \le n,\ i \ne j)$ $\quad x_i \longmapsto x_j x_i x_j^{-1}$

NORMALLY GENERATES AS SUBGROUP OF $\mathrm{Aut}(F_n)$

ii) $m_{ijk}$ $(1 \le i, j, k \le n$ DISTINCT$)$ $\quad x_i \longmapsto x_i [x_j, x_k]$

(NOT AN OBVIOUS THEOREM, STILL TODAY...)

JOHNSON HOMOMORPHISM: $\tau: IA_n \longrightarrow \mathrm{Hom}(\mathbb{Z}^n, \Lambda^2 \mathbb{Z}^n)$

$\varphi: [F_n, F_n] \longrightarrow \Lambda^2 \mathbb{Z}^2$ QUOTIENT BY CENTRAL SERIES

For $f \in IA_n$, $x \in F_n$,

$\tau(f)([x]) = \varphi(f(x) x^{-1})$ FUN EX: it is a well-defined homomorphism.

OBSERVATION: $\tau$ TAKES $X_n^{IA}$ TO ~~BASIS~~ A BASIS OF THE ABELIAN GROUP $\mathrm{Hom}(\mathbb{Z}^n, \Lambda^2 \mathbb{Z}^n)$

eg: $\tau(m_{ijk}): [x_i] \longmapsto [x_j] \wedge [x_k]$

$\tau(C_{ij}): [x_i] \longmapsto [x_i] \wedge [x_j]$

EXERCISE: THIS IMPLIES COR (FARB, COHEN-PAKIANATHAN, KAWAZUMI)

$H_1(IA_n) = \mathrm{Hom}(\mathbb{Z}^n, \Lambda^2 \mathbb{Z}^n)$ AS $GL_n^{\mathbb{Z}}$-REP

HARD OPEN QUESTION: IS $H_2(IA_n)$ FINITELY GEN?

THM (KRSTIĆ-McCOOL '97 + BESTVINA-BUX-MARGALIT '07)
$H_2(IA_3, \mathbb{Q})$ IS $\infty$-DIM'L.

$GL_n\mathbb{Z} \curvearrowright H_*(IA_n)$

THM (DP) $\exists$ AN EXPLICIT FINITE SET $I_n \subset H_2(IA_n)$ s.t. $GL_n\mathbb{Z} \cdot I_n$ SPANS $H_2(IA_n)$, i.e. IT IS FINITELY GENERATED AS A $GL_n\mathbb{Z}$-REP.

$I_n$ is "THE SAME" FOR ALL $n \geq 3$.

COR (SURJECTIVE REPR. STABILITY): FOR $n \geq 3$, THE $GL_{n+1}\mathbb{Z}$ ORBIT OF $\mathrm{Im}(H_2(IA_n) \to H_2(IA_{n+1}))$ SPANS.
IN FACT $\mathrm{Im}(H_2(IA_3)) \to H_2(IA_n))$

(SIMILAR THM FOR TORELLI GROUP BY BOLDSEN-DOLLERUP.)

COR: FOR $n \geq 3$, $(H_2(IA_n))_{GL_n\mathbb{Z}} = 0$.

DEF: $\mathrm{Aut}(F_n)[\ell] = \ker(\mathrm{Aut}(F_n) \to GL_n(\mathbb{Z}/\ell))$

COR: FOR $n \geq 3$, $\ell \geq 2$, $H_2(\mathrm{Aut}(F_n)[\ell]; \mathbb{Q}) = 0$

(BOREL$^{\text{SIMILAR BY}}$ FOR $GL_n$, PUTMAN FOR MCG)

# IDEA OF PROOF OF THE THM

HOPF: $G = F(S)/R$ THEN $H_2(G) = \dfrac{R \cap [F(S), F(S)]}{[F(S), R]}$

FREE GROUP ON A SET S

NORMAL SUBGROUP OF RELATIONS

PROBLEM: THE INTERSECTION IS DIFFICULT TO COMPUTE...

ASSUME THAT WE HAVE A PRESENTATION

$$IA_n = \langle X_n^{IA} \mid R_n^{IA} \rangle$$

THEN $R_n^{IA} \subseteq [F(X_n^{IA}), F(X_n^{IA})]$

$\Rightarrow H_2(IA_n) = \dfrac{\langle\langle R_n^{IA} \rangle\rangle}{\text{SOMETHING}}$ — NORMAL CLOSURE

$\Rightarrow$ GET GENERATORS FOR $H_2(IA_n)$ IN BIJECTION WITH $R_n^{IA}$.

$$r = [a_1, b_1] \dots [a_g, b_g] \longleftrightarrow \varphi_r : \Sigma_g \longrightarrow K(IA_n, 1)$$

$r \in R_n^{IA}$ $\qquad \varphi_r([\Sigma_g]) \in H_2(IA_n)$

BUT WE DON'T HAVE SUCH A PRESENTATION...

~~DEF~~: (BARTHOLDI) A FINITE L-PRESENTATION FOR A GROUP G IS:
- A GENERATING SET S
- A FINITE SET $R_0 \subseteq F(S)$
- A FINITE SET $E \subseteq \mathrm{End}(F(S))$

S.T. THE FOLLOWING HOLDS: LET $\Gamma \subseteq \mathrm{End}(F(S))$ BE THE MONOID GENERATED BY E. SET

$R = \{ f(r) \mid f \in \Gamma, r \in R_0 \}$

THEN $G = \langle S \mid R \rangle$

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PICK A GENERATING SET $X_n^{AF}$ FOR $Aut(F_n)$.

HAVE A PROJECTION $\pi : F(X_n^{IA}) \twoheadrightarrow IA_n$

FOR $f \in X_n^{AF}$, HAVE CONJUGATION ACTION

$i_f \in Aut(IA_n)$. LET $e_f \in End(F(X_n^{IA}))$ BE ANY

LIFT OF $i_f$. [NOTE: THIS APPEARS ALREAD IN MAGNUS'S PAPER/WORK]

$E_n^{IA} = \{ e_f \mid f \in X_n^{AF} \}$

GOAL: FIND SOME $R_n^{IA} \subseteq F(X_n^{IA})$ s.t. $IA_n$ HAS

L-PRESENTATION $\langle X_n^{IA} \mid R_n^{IA} \mid E_n^{IA} \rangle$

THEN $GL_n\mathbb{Z}$-ORBITS OF ELEMENTS OF $H_2(IA_n)$

ASSOCIATED TO $R_n^{IA}$ will SPAN.

GUESS THE RELATIONS $R_n^{IA}$:

LET $\Gamma_n$ = GROUP WITH PURPORTED L-PRES.

FOLLOWING IS EASY: VERIFY THAT $Aut(F_n)$ ACTS ON

$\Gamma_n$, LIFTING ACTION ON $IA_n$ THROUGH $\Gamma_n \twoheadrightarrow IA_n$

(RELATIONS BUILD SO THAT THIS IS EASY)

ALLOWS US TO USE TECHNIQUES INTRODUCED BY BRENDLE-MARG-P

TO STUDY THE KER OF THE BURAU REPR AT $-1$.

USES 2 OTHER TOOLS: ANALOGUE OF COMPLEX OF CURVES (COMPLEX OF PARTIAL BASES)

ANALOGUE OF BIRMAN EXACT SEQUENCE.