H_*-STABILITY AND MODULI SPACES

PART 1 : SØREN GALATIUS

DEF: FIX $P^{m-1} \subset \mathbb{R}^\infty$ CLOSED SMOOTH (POSSIBLY $\emptyset$)

$$\mathcal{N}(P) = \left\{ W^m \subset (-\infty, 0] \times \mathbb{R}^\infty \;\middle|\; \begin{array}{l} \text{COMPACT + SMOOTH} \\ \partial W = \{0\} \times P \\ \exists \varepsilon\ (-\varepsilon, 0] \times P \subset W \end{array} \right\}$$

$$\simeq \coprod_{\substack{W, \\ \partial W = P}} BDiff(W, \partial W)$$

$H^0(\mathcal{N}(P))$ = INVARIANTS OF MANIFOLDS

$H^i(\mathcal{N}(P))$ = CHARACTERISTIC CLASSES OF BDIS
$i > 0$

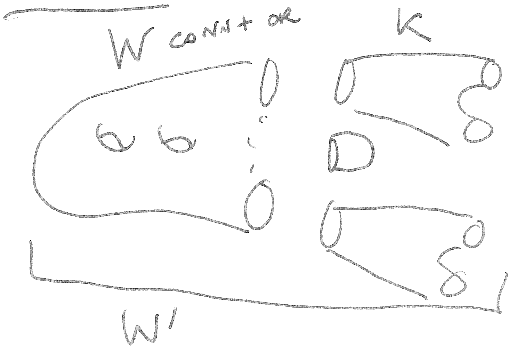
TOOL: HOMOLOGICAL STABILITY

GIVEN A BORDISM $K : P \rightsquigarrow Q$ in $[0,1] \times \mathbb{R}^\infty$, GET A CONT. MAP

$$\mathcal{N}(P) \longrightarrow \mathcal{N}(Q)$$
$$W \longmapsto (W \cup K) - e_1$$

SUBSTRACT 1 TO THE FIRST COORD.

THM (HARER): $m = 2$



$$BDiff^{or}(W, \partial) \longrightarrow BDiff^{or}(W', \partial)$$

W, W' CONNECTED

HARER: ISO FOR H^*, $* \ll g$ IN STABLE RANGE

MADSEN-WEISS : $H^*(-; \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \ldots]$

$m = 2n$

Def: $W^{2n}_{g,1} = \#_g S^n \times S^n \setminus \mathrm{int}(D^{2n})$

FOR W^{2n} CONNECTED, DEFINE ITS GENUS:

$$g(W) = \max\{g \mid \exists\, W_{g,1} \hookrightarrow W\}$$

NOTE: $g(W \# (S^n \times S^n)) \geq g(W) + 1$ (DOESN'T NEED TO HAVE EQUALITY HERE)

$$\bar{g}(W) = \max\{g(W \# k(S^n \times S^n)) - k \mid k\}$$

"STABLE GENUS"

THEN $\bar{g}(W \# (S^n \times S^n)) = \bar{g}(W) + 1$

RESULTS:

DESCRIPTION OF $H_i(B\mathrm{Diff}(W, \partial W))$ FOR $n > 2$, $\pi_1 W = 0$

FOR $* \leq \frac{\bar{g}(W) - 4}{2}$

FALSE: $B\mathrm{Diff}(W, \partial) \longrightarrow B\mathrm{Diff}(W', \partial)$ ISO IN H_* IN A RANGE

$W' = W \cup K$

— AT LEAST NOT ALWAYS!

TANGENTIAL n-TYPE OF W^{2n}

$$\begin{array}{c} X \\ \downarrow \theta \\ BO(2n) \end{array}$$

IS SERRE FIBRATION

n-COCONNECTED (i.e. $\pi_{\geq n}(\text{FIBER}) = 0$)

IF $\exists$ $W \xrightarrow{\ell} X \xrightarrow{\theta} BO(2n)$, $W \xrightarrow{TW} BO(2n)$ s.t. ℓ is n-CONNECTED

TW ← CLASSIFYING MAP OF $TW \to W$.

CLASSICAL FACT (MOORE-POSTNIKOV FACTORISATION)
$\forall\ W \xrightarrow[TW]{} BO(2n)$, $\exists$ SUCH AN (X, θ, ℓ).

INSTEAD OF CLASSIFYING ALL MANIFOLDS, FIX
$X \xrightarrow{\theta} BO(2n)$, AND CLASSIFY PAIRS (W, ℓ).

$$\mathcal{N}^\theta(P, \ell_P) := \left\{ (W, \ell_W) \;\middle|\; \begin{array}{l} W \text{ AS BEFORE} \\ \ell_W : \begin{array}{ccc} TW & \to & \theta^*\gamma \\ \downarrow & & \downarrow \\ W & \to & X \end{array} , \end{array} \begin{array}{l} \ell_W\ n\text{-CONN}, \\ \ell_W|_{\partial W} = \ell_P \end{array} \right\}$$

FIX
$$\begin{array}{ccc} \varepsilon^1 \oplus TP & \to & \theta^*\gamma \\ \downarrow & & \downarrow \\ P & \longrightarrow & X \end{array}$$

LIFT OF $P \longrightarrow BO(2n)$
CLASSIFYING $\varepsilon^1 \oplus TP$
($\gamma \to BO(2n)$ CANONICAL BDL)

GIVEN A BORDISM $P \overset{K}{\rightsquigarrow} Q$, TOGETHER WITH

$$\begin{array}{ccc} TK & \xrightarrow{\ell_K} & \theta^*\gamma \\ \downarrow & & \downarrow \\ K & \longrightarrow & X \end{array} \qquad \mathcal{N}^\theta(P) \xrightarrow{\cup_P K} \mathcal{N}^\theta(Q)$$

WELL-DEFINED IF $P \hookrightarrow K$ IS $(n-1)$-CONNECTED.

THIS IS THE RIGHT KIND OF STABILIZATION MAP.

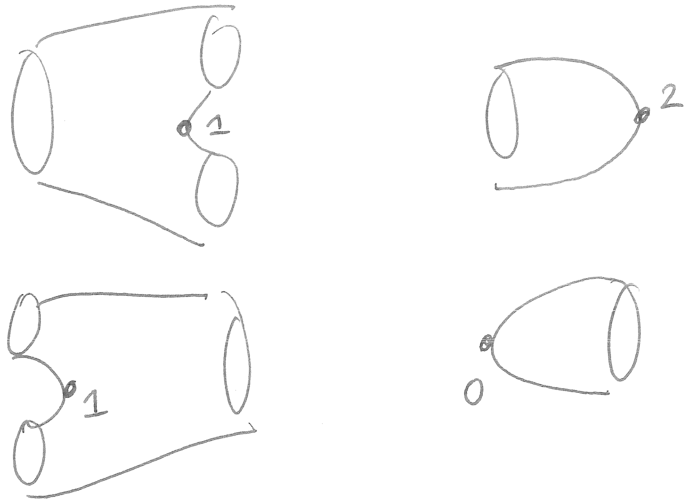
(ASSUME X CONNECTED)

REQUIRE $\bar{g}(W) = g$

$$\mathcal{N}^\theta(P) = \coprod_{g \geq 0} \mathcal{N}^{\theta, g}(P)$$

<u>ELEMENTARY BORDISMS</u> : $K : P \to Q$ OF INDEX k IS ELEMENTARY IF IT ADMITS A MORSE FUNCTION $f : K \to [0,1]$, $f^{-1}(0) = P$, $f^{-1}(1) = Q$, EXACTLY ONE CRIT POINT OF INDEX k.

EX:



It suffices to consider elementary bordisms of index $k \in [n, 2n]$.

If $k > n$,

$$\begin{tikzcd}
\mathcal{N}^{\theta,g}(P) \arrow[r, "K"] \arrow[dr, "K\bar{K}"'] & \mathcal{N}^{\theta,g}(Q) \\
 & \mathcal{N}^{\theta,g+1}(P)
\end{tikzcd}$$

<u>THM</u>: Both maps induce iso in H_*, $* \le \frac{g-4}{2}$

if $n > 2$, $\pi_1 X = 0$.

Special case:

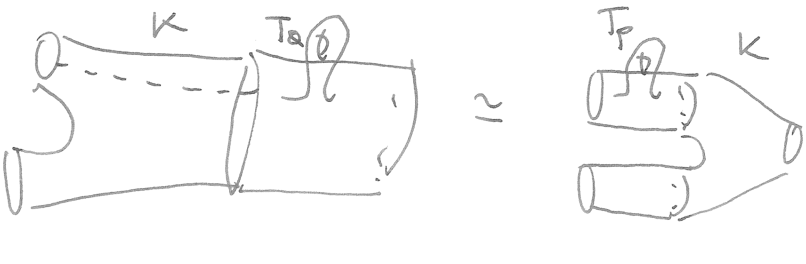
<u>THM A</u>: $T_P = ([0,1] \times P) \# S^n \times S^n$, then "unstable stability"

$\mathcal{N}^{\theta,g-1}(P) \longrightarrow \mathcal{N}^{\theta,g}(P)$ induces an iso

in H_*, $* \le \frac{g-4}{2}$

epi, $* \le \frac{g-4}{2} + 1$

$\longrightarrow$ Then enough to prove "stable stability" for other bordisms (wrt T)



For $k \in [n, 2n-1]$

$T_Q \circ K \cong K \circ T_P$

$$\begin{tikzcd} \mathcal{N}^\theta(P) \arrow[r, "T_P"] \arrow[d, "K"] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{N}^\theta(P) \arrow[r] \arrow[d, "K"] & \cdots \\ \mathcal{N}^\theta(Q) \arrow[r, "T_Q"] & \mathcal{N}^\theta(Q) \arrow[r] & \cdots \end{tikzcd}$$

GIVEN THM A, IT SUFFICES FOR $k \in [n, 2n-1]$ TO PROVE

<u>THM B</u>: $H_*(\mathcal{N}^\theta(P)) \xrightarrow{K_*} H_*(\mathcal{N}^\theta(Q))$ is $\mathbb{Z}[T]$-LINEAR. IT is AN ISOMORPHISM AFTER INVERTING T (i.e. AFTER $\underset{\mathbb{Z}[T]}{\otimes} \mathbb{Z}[T, T^{-1}]$)

(THIS LAST THM B ALSO TRUE WHEN $n=2$, $\pi_1 X \neq 0$).

<u>THM</u>: $\mathcal{N}^\theta(P) \longrightarrow \Omega^\infty X^{-\theta}$ ISO IN H_* IN STABLE RANGE

$$\underset{\mathrm{Aut}(\theta)}{\circlearrowleft} \qquad\qquad \underset{\mathrm{Aut}(\theta)}{\circlearrowleft}$$

$$\rightsquigarrow H_*(B\mathrm{Diff}(W)) \underset{\text{STABLE RANGE}}{\cong} H_*((\Omega^\infty X^{-\theta})_{h\mathrm{Aut}(\theta)})$$

PART II: OSCAR RANDAL-WILLIAMS

SLIGHT VARIANT IN THE MODEL FOR $\mathcal{N}^{\theta}(P)$:

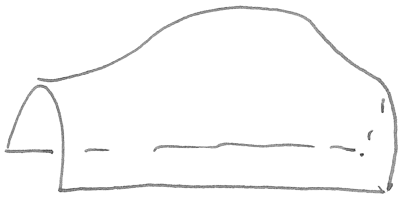
$P \subset \mathbb{R}^\infty$, CHOOSE $S^{2n-1} \subset \mathbb{R}^{2n} \subset \mathbb{R}^\infty$, $L = D_-^{2n-1} \subset S^{2n-1}$,

$\ell_{S^{2n-1}} : \varepsilon^1 \oplus TS^{2n-1} \to \theta^*\gamma$

$\mathcal{N}^\theta(P, \ell_P) = \Big\{$  , $\ell_W : TW \longrightarrow \theta^*\gamma \Big\}$

WHICH IS STANDARD ON P, $[0,1]\times L$, S^{2n-1}

P
$\downarrow$
$P \setminus L$



$\in \;\simeq \big(\mathcal{N}^\theta(P, \ell_P)$ IN SØREN'S TALK$\big)$

$$T = \left([0,1] \times S^{2n-1}\right) \# \left(S^n \times S^n\right) \subset [0,1] \times \mathbb{R}^\infty$$

COMPOSITION ON THE RIGHT BY T GIVES AN ENDOMORPHISM

$$- \circ T : \mathcal{N}^\theta(P) \longrightarrow \mathcal{N}^\theta(P)$$

LET $\mathcal{N}^\theta(P, \ell_P)[T^{-1}]$ BE THE MAPPING TELESCOPE OF THIS MAP.

LET $\mathcal{C}^{n-1}$ BE THE 'CATEGORY' WITH
MISSING IDENTITIES

OBJECTS: (P, ℓ_P), $P \subset \mathbb{R}^\infty$ AND CONTAINS L

MORPH: COBORDISMS $(K, \ell_K) : (P, \ell_P) \rightsquigarrow (Q, \ell_Q)$
INSIDE $[0,s] \times \mathbb{R}^\infty$ THAT CONTAIN $[0,s] \times L$ WHICH ARE $(n-1)$-CONN REL Q.

CAN TOPOLOGIZE ...

$N^\theta(-)$ IS A FUNCTOR: $\mathcal{C}^{n-1} \longrightarrow Top$

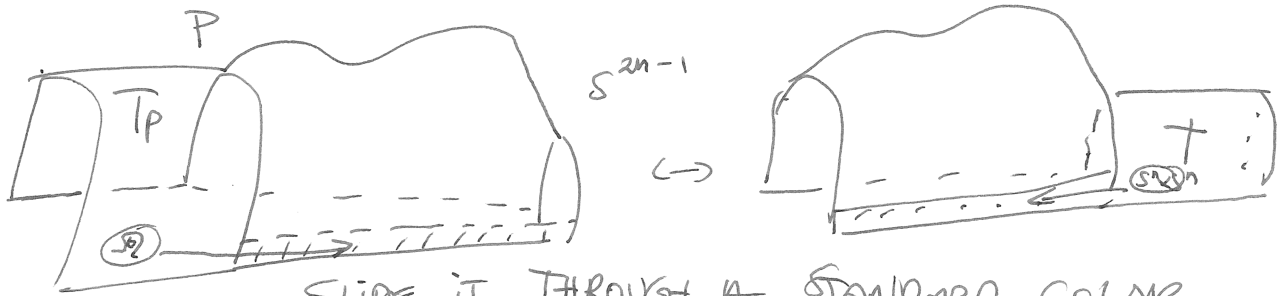
$$(P, \ell_P) \longmapsto N^\theta(P, \ell_P)$$
$$(K, \ell_K) \longmapsto (K \cup -) + e_1$$

NOTE: WE NOW GLUE T ON A DIFFERENT SIDE THAN K $\Rightarrow$ THESE OPERATIONS STRICTLY COMMUTE: $(- \circ T)$ GIVES A SELF NAT TRANSF. OF $N^\theta(-)$, SO GET A FUNCTOR $N^\theta(-)[T^{-1}]: \mathcal{C}^{n-1} \longrightarrow Top$

THM (STABLE STABILITY) IF $2n \geq 4$, THIS SENDS EVERY MORPHISM IN $\mathcal{C}^{n-1}$ TO A H_*-EQUIV.

OBSERVATION: $T_P = ([0,1] \times P) \# (S^n \times S^n)$, $T_P : N^\theta(P, \ell_P)[T^{-1}]$ ↺

BUT

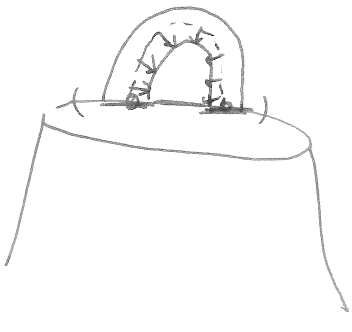


SLIDE IT THROUGH A STANDARD COLLAR

$\rightarrow T_P$ INDUCES AN ISO ON H_* OF $N^\theta(P)[T^{-1}]$

THE COBORDISM T_P HAS THE FORM $\bar{M} \circ M$ WHERE $M: P \rightsquigarrow Q$ CONSISTS OF A SINGLE n-HANDLE, ATTACHED TRIVIALLY

EX:

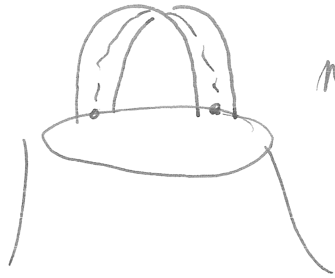


THE BOUNDARY OF THE CORE BOUNDS A DISC IN ∂, + NORMAL BDL... (COMPATIBLE FRAMING)

NON-TRIVIAL ATTACHMENTS :

NO DISC INTO WITH BDRY THE BDRY OF THE CORE

OR



NOT COMPATIBLE FRAMING

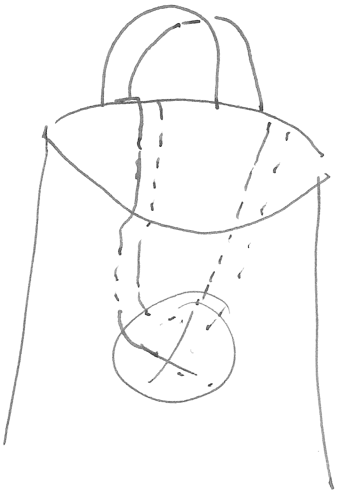
IF $M : P \rightsquigarrow Q$ IS A COBORDISM CONSISTING OF A SINGLE TRIVIALLY ATTACHED HANDLE, THEN $\bar{M} \circ M \simeq T_P$ SO $\bar{M} \circ M$ INDUCES A H_*-ISO.

IDEA: MODIFY THE MFD TO MAKE THE ATTACHMENT TRIVIAL.

EX:

FIND A 1-HANDLE AND CUT IT OUT

→ NOW THE ATTACHMENT IS TRIVIAL



⇝ AGAIN TRIVIAL AFTER CUTTING OUT.

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GIVEN $M: P \rightsquigarrow Q$, CONSIDER $\bar{M} \circ M : N^\theta(P)[T^{-1}] \circlearrowleft$

WANT TO CONSIDER A SEMI-SIMPLICIAL RESOLUTION

$$K_\bullet(P) \xrightarrow{\varepsilon} N^\theta(P) \quad \text{s.t.}$$

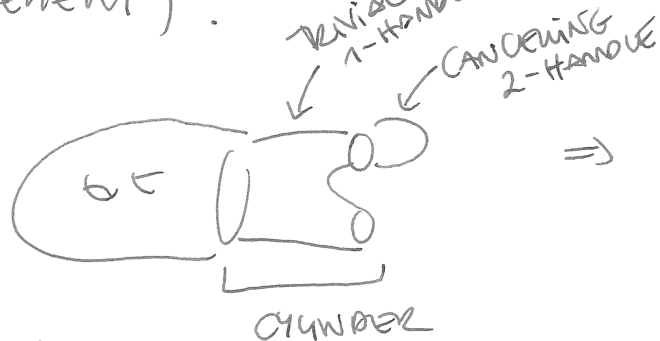
1) $K_p(P) \simeq N^\theta(R)$ FOR SOME R

2) $(\bar{M} \circ M)$ LIFTS TO A SEMI-SIMPLICIAL MAP, WHICH IN deg p IS GIVEN BY $\bar{M}' \circ M'$, WHERE M' CONSISTS OF A TRIVIAL k-HANDLE.

3) THE MAP $|\varepsilon| : |K_\bullet(P)| \xrightarrow{\sim} N^\theta(P)$

$K_p(P)$ IS THE SET OF CHOICES OF $p+1$ HANDLES MAKING THE ATTACHMENT TRIVIAL (CORRESPONDS TO, IN THE NON-ORIENTABLE SURFACE CASE TO COLLECTIONS OF 1-SIDED ARCS WITH NON-ORIENTABLE, CONNECTED COMPLEMENT).

NOTE:



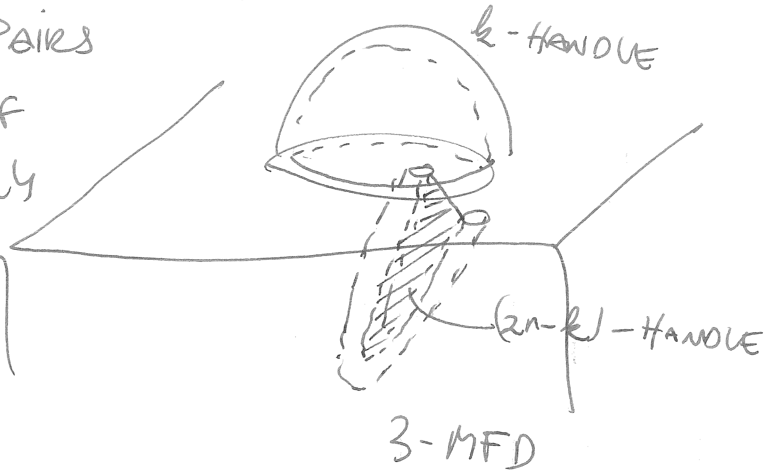
$\Rightarrow$ THE FIRST ONE INDUCES INJECTIVE MAP IN H_* AND THE SECOND A SURJ. MAP (IF $\exists$ ANOTHER BDRY.)

MORE GENERALLY,

IF $M: P \rightsquigarrow Q$ CONSISTS OF A TRIVIAL $(k-1)$-HANDLE, AND $N: R \rightsquigarrow P$ IS A CANCELLING k-HANDLE, THEN $M \circ N : R \rightsquigarrow Q$ IS DIFFEO TO A CYLINDER

STABILITY FOR $(k-1)$-HANDLES $\Rightarrow$ STAB FOR CANCELLING k-HANDLES

MAKE A RESOLUTION $L_\bullet(P) \to N^\sigma(P, \ell_P)$
WHERE 0-SIMPLICES ARE PAIRS
(W, ℓ_W) WITH EMBEDDING OF
A $(2n-k)$-HANDLE WITH BDRY
A MERIDIAN OF THE HANDLE
ATTACHED VIA M



$$
\begin{tikzcd}
0 & \mathbb{Z} \arrow[r, shift left, "id"] \arrow[r, shift right, "-id"'] & \mathbb{Z}_{a-b} \arrow[r, shift left] \arrow[r] \arrow[r, shift right] & \mathbb{Z}/2 \arrow[r, "\frac{a+b+c}{\mathrm{mod}\ 2}"] & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \arrow[r] & \cdots \\
0 \arrow[u] & \mathbb{Z} \arrow[u, "\cong"] \arrow[r, shift left, "(1,0)"] \arrow[r, shift right, "(0,1)"'] & \mathbb{Z}^2 \arrow[u] \arrow[r, shift left, "(a,b,0)"] \arrow[r, "(a,0,b)"] \arrow[r, shift right, "(a,b,0)"'] & \mathbb{Z}^3 \arrow[u] \arrow[r] & \mathbb{Z}^4 \arrow[u] \arrow[r] & \mathbb{Z}^5 \arrow[u] \arrow[r] & \cdots \\
0 & 0 \arrow[r] & \mathbb{Z} \arrow[u, "\Delta"] \arrow[loop below, "\mathrm{trivial}"] \arrow[r, shift left] \arrow[r] \arrow[r, shift right] & \mathbb{Z}^3_{ev} \arrow[u] \arrow[r] & \mathbb{Z}^4_{ev} \arrow[r] & \mathbb{Z}^5_{ev} \arrow[r] & \cdots
\end{tikzcd}
$$

$(a,b) \mapsto (a,b,0)$ $\qquad$ (a,b,c)

$\mathbb{Z}^3_{ev} = \{(a,b,c) \mid a+b+c \text{ even}\}$

<u>GENERATORS</u>: $e_1+e_2 \in V_2$
$= (1,1)$

$2e_1 \in V_3$
$= (2,0,0)$

REDUNDANT

BUT $2e_1 = (e_1+e_2)+(e_1+e_3) - (e_2+e_3)$

$$
\begin{array}{c}
0 \\ \downarrow \\ W \\ \downarrow \\ M(\mathrm{trm}_2) = \mathbb{Z}\left[\binom{S}{2}\right] \\ \downarrow \\ V
\end{array}
$$

$$
\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}^3 \arrow[r] \arrow[d] & \mathbb{Z}^6 \arrow[r] \arrow[d] & \mathbb{Z}^{10} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}^3_{ev} \arrow[r] & \mathbb{Z}^4_{ev} \arrow[r] & \mathbb{Z}^5_{ev} \arrow[r] & \cdots
\end{tikzcd}
$$

FIND GENERATORS FOR W

<u>H_* - STAB AND MODULI SPACES</u>