

AREWEN DILBERT : GENERALIZED POLYNOMIAL FUNCTORS

(2 RECENT PREPRINTS, ONE WITH C. VEDRA)

A RING. $S(A)$: CATEGORY OF f.g FREE A -MODULES WITH SPLITTED INJECTIONS.

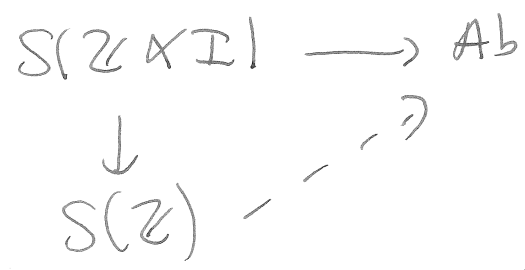
IF I IS A RING WITHOUT UNIT, WE CAN ADD FORMALLY A UNIT TO $I \rightsquigarrow \mathbb{Z} \rtimes I \leftarrow \mathbb{Z}$

EX: $\begin{matrix} I \\ \subset \mathbb{Z} \times \mathbb{Z} \\ \supset \mathbb{Z} \subset \mathbb{Z} \end{matrix}$
CONGRUENCE SUBGROUP OF OTHER TALKS

THE CONGRUENCE GENERAL LINEAR GROUP FOR I , i.e.

$$GL_n(I) := \ker (GL_n(\mathbb{Z} \rtimes I) \xrightarrow{\quad} GL_n(\mathbb{Z}))$$

WE CAN SEE $H_n(GL_n(I))$ AS A FUNCTOR ON $S(\mathbb{Z})$.



CONJECTURE: For all RINGS WITHOUT UNIT I AND ALL $n \in \mathbb{N}$, THE FUNCTOR $H_n(GL_n(I)): S(\mathbb{Z}) \rightarrow Ab$ IS A WEAK POLYNOMIAL FUNCTOR.

NOTE: IT'S PROBABLY FALSE FOR STRONG POLYNOMIAL FUNCTORS WITHOUT SOME BARS CONDITION ON A .

ANOTHER AIM: STUDY $\text{colim}_n H_n(GL_n(I); F(I^n))$

↑ POLYNOMIAL DOES NOT FIT IN THE SET-UP OF CHRISTINE'S TALK BECAUSE NOT THE AUT OF SOME OBJ IN A SYMM MON CAT.

OTHER MOTIVATION: GROUPS $IA_n := \ker(\text{Aut}(F_n) \rightarrow GL_n(\mathbb{Z}))$

QUESTION: (i) WHAT ^{DOES} $H_*(IA_n)$ LOOK LIKE?

(ii) WHAT DOES $\gamma_2(IA_n) / \gamma_{2+1}(IA_n)$ LOOK LIKE?

LOWER CENTRAL
SERIES

THIS IS A POLYNOMIAL FUNCTOR
(\Leftrightarrow FINITELY GEN. FI-MODULE)

THESE CAN BE SEEN AS FUNCTORS $S(\mathbb{Z}) \xrightarrow{\exists} \text{Ab}$
 \uparrow ABELIANIZATION
 G

STRONG POLYNOMIAL FUNCTORS

$(\mathcal{M}, +, 0)$ SMALL SYMM. MONOIDAL CATEGORY WITH THE UNIT 0 IS INITIAL.

\mathcal{A} = ABELIAN CATEGORIES.

IN $\text{Fct}(\mathcal{M}, \mathcal{A})$, WE HAVE ENDO-FUNCTORS τ_n ,
 FOR $n \in \text{Obj}(\mathcal{M})$ PRECOMPOSITION BY " n + "
 BY " n + "

WE HAVE AN OBVIOUS NAT TRANS. $\tau_0 = \text{Id} \rightarrow \tau_n$ -
 ITS KERNEL IS DENOTED δ_n AND IS RIGHT EXACT.
 (KERNEL δ_n)

DEF: $F: \mathcal{M} \rightarrow \mathcal{A}$ IS SAID TO BE STRONGLY POLYNOMIAL
 OF STRONG DEGREE $\leq d$ IF $\forall (a_0, \dots, a_d) \in \text{Obj}(\mathcal{M}^{d+1})$,
 $\delta_{a_0} \dots \delta_{a_d} F = 0$.

EX: $\mathcal{M} = \text{FI}$, $\mathcal{A} = \text{Ab}$. F IS STRONGLY POLYNOMIAL
 AND WITH VALUES IN \mathbb{Z} OF ABELIAN GPS $\Leftrightarrow F$ IS f.g AS
 FI-MODULE.

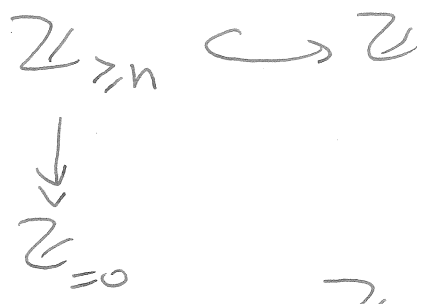
PROBLEM: THIS NOTION DOES NOT BEHAVE WELL ~~UNDER~~ RELATIVELY TO SUBOBJECTS.

EX: $M = FI$ \mathbb{Z} POLY STRONG DEGREE 0

$A = Ab$

$\forall n \in \mathbb{N}, \mathbb{Z}_{\geq n} : E \mapsto \begin{cases} \mathbb{Z} & \text{if } |E| \geq n \\ 0 & \text{else} \end{cases}$

$\mathbb{Z}_{=n} : E \mapsto \begin{cases} \mathbb{Z} & |E| = n \\ 0 & \text{else} \end{cases}$



$\delta_n \mathbb{Z}_{\geq n} \simeq \mathbb{Z}_{=n-1}$

$\delta_n \mathbb{Z}_i \simeq \mathbb{Z}_{i-1}$

$\mathbb{Z}_{\geq n}$ AND $\mathbb{Z}_{=n}$ ARE POLYNOMIAL OF STRONG DEGREE n .

WEAK POLYNOMIAL FUNCTORS

DEF: WE DENOTE $SN(\mathcal{M}, A)$ THE FULL SUBCAT OF $Fct(\mathcal{M}, A)$ OF FUNCTORS S.T. $F = \sum_{n \in Obj(\mathcal{M})} k_n(F)$

(EX: IN FI-MODULUS, THIS IS EQUIVALENT TO $\text{cod} F(n) = 0$ $n \in \mathbb{N}$)

$SN(\mathcal{M}, A)$ IS A LOCALIZING SUBCATEGORY OF $Fct(\mathcal{M}, A)$, THICK AND STABLE UNDER COLIMITS.

$\rightarrow St(\mathcal{M}, A) := Fct(\mathcal{M}, A) / SN(\mathcal{M}, A)$

THE FUNCTORS τ_n STILL INDUCE EXACT ENDOFUNCTORS OF $\text{St}(M, A)$. $\rightarrow \delta_n = \text{coker}(\text{Id} \rightarrow \tau_n)$ IN $\text{St}(M, A)$

↑
HAS BECOME MONO

⇐
 δ_n IS EXACT.

DEF: (i) AN OBJECT X OF $\text{St}(M, A)$ IS SAID POLYNOMIAL OF DEGREE $\leq d$ IF $\forall a_0, \dots, a_d, \delta_{a_0} \dots \delta_{a_d}(X) = 0$

(ii) A FUNCTOR $F: M \rightarrow A$ IS WEAKLY POLYNOMIAL IF ITS IMAGE IN $\text{St}(M, A)$ IS POLY OF DEG $\leq d$.

(NOTE: THE BAD EXAMPLE FROM ABOVE DISAPPEARED.)

Denote $\text{Pol}_d(M, A)$ THEIR FULL SUBCAT.

FACT: \hookrightarrow IS A BICOENRICHING SUBCATEGORY OF $\text{St}(M, A)$.

WE CAN FORM $\text{Pol}_d(M, A) / \text{Pol}_{d-1}(M, A)$

MAIN RESULT OF (D-V): DESCRIPTION OF $\text{Pol}_d(M, A) / \text{Pol}_{d-1}(M, A)$ WHEN

$M = \text{CATEGORY OF HAMILTONIAN OBJECTS (FOR EX } \mathbb{S}(A) \text{)}$

EX: $\text{Pol}_0(M, A) = A$ (\equiv CONSTANT FUNCTORS)

• $\text{Pol}_d(\mathbb{S}(\mathbb{Z}), \text{Ab}) / \text{Pol}_{d-1} \cong \mathbb{Z}[\mathbb{S}_d]$ -BIMODULES
↑
SYMM GROUPS

CHURCH-TURNER-BERG-FORB-NAGPAL:

$\text{FCT}(FI, A)$ IS LOCALLY NOETHERIAN

↳ LOCALLY NOETHERIAN CAT

(i.e. \exists SET OF NOETHERIAN GENERATORS)

CONSEQUENCE: UNDER MILD ASSUMPTIONS ON M

(e.g. $M = S(A)$) IF $F: M \rightarrow A$ IS SOMEHOW POLYNOMIAL AND TAKEER VIEWER IN NOETHERIAN OBJECTS OF A , THEN F IS NOETHERIAN.

THM (i) IF A IS A FINITE RING, THEN $\text{Pol}(S(A), Ab)$ IS LOCALLY NOETHERIAN

(ii) $\text{Pol}(S(\mathbb{Z}), Ab)$ IS "ALMOST" LOCALLY NOETHERIAN
↳ $\mathbb{Z}[G_n(\mathbb{Z})]$ NOT LOCALLY NOETHERIAN...

General Construction:

$$M \xrightarrow{\sim} \tilde{M}$$

o INITIAL o IS NUL

eg $FI \xrightarrow{\sim} FI\#$

$$\text{Pol}_d(M, A) / \text{Pol}_{d-1} \cong \text{Pol}_d(\tilde{M}, A) / \text{Pol}_{d-1}$$