Aurélien Djament: Generalized polynomial functors

(2 recent preprints, one with C. Vespa)

A ring. $S(A)$ = category of f.g. free A-modules with splitted injections.

If I is a ring without unit, we can add formally a unit to I ⇝ $\mathbb{Z} \ltimes I \rightleftarrows \mathbb{Z}$

[ex: $\mathbb{Z} \ltimes \overset{I}{p\mathbb{Z}}$
$\subset$
$p\mathbb{Z} \subset \mathbb{Z}$
⇝ congruence subgroup of other talks]

The congruence general linear group for I, i.e.

$$GL_n(I) := \ker\left(GL_n(\mathbb{Z} \ltimes I) \rightleftarrows GL_n(\mathbb{Z})\right)$$

We can see $H_*(GL_n(I))$ as a functor on $S(\mathbb{Z})$.

$$\begin{tikzcd} S(\mathbb{Z} \ltimes I) \arrow[r] \arrow[d] & Ab \\ S(\mathbb{Z}) \arrow[ur, dashed, "?"'] & \end{tikzcd}$$

<u>Conjecture</u>: For all rings without unit I and all $n \in \mathbb{N}$, the functor $H_n(GL(I)) : S(\mathbb{Z}) \to Ab$ is a <u>weak</u> polynomial functor.

<u>Note</u>: It's probably false for strong polynomial functors without some base condition on A.

<u>Another aim</u>: study $\operatorname{colim}_n H_*(GL_n(I); F(I^n))$

F ↳ polynomial

$GL_n(I)$ ↑ does not fit in the set-up of Christine's talk because not the aut of some obj in a <u>symm</u> mon cat.

OTHER MOTIVATION: GROUPS $IA_n := \ker(\mathrm{Aut}(F_n) \twoheadrightarrow GL_n(\mathbb{Z}))$

QUESTIONS: (i) WHAT DOES $H_*(IA_n)$ LOOK LIKE?

(ii) WHAT DOES $\underbrace{\gamma_\ell(IA_n)/\gamma_{\ell+1}(IA_n)}$ LOOK LIKE?

LOWER CENTRAL SERIES

THIS IS A POLYNOMIAL FUNCTOR
($\Leftrightarrow$ FINITELY GEN FI-MODULE)

THESE CAN BE SEEN AS FUNCTORS

$$\begin{tikzcd} & S(\mathbb{Z}) \arrow[r, "\exists"] & Ab \\ & \mathcal{G} \arrow[u, "\text{ABELIANIZATION}"] \arrow[ur] & \end{tikzcd}$$

STRONG POLYNOMIAL FUNCTORS

$(\mathcal{M}, +, 0)$ SMALL SYMM. MONOIDAL CATEGORY WITH THE UNIT 0 IS INITIAL.

$\mathcal{A}$ = ABELIAN CATEGORY.

IN $Fct(\mathcal{M}, \mathcal{A})$, WE HAVE ENDO-FUNCTORS τ_n, FOR $n \in Obj(\mathcal{M})$

τ_n = PRECOMPOSITION BY "$n + \_$"

WE HAVE AN OBVIOUS NAT TRANSF. $\tau_0 = Id \longrightarrow \tau_n$.
ITS COKERNEL IS DENOTED δ_n AND IS RIGHT EXACT.
(KERNEL κ_n)

Def: $F : \mathcal{M} \to \mathcal{A}$ is SAID TO BE STRONGLY POLYNOMIAL OF STRONG DEGREE $\leq d$ if $\forall (a_0, \dots, a_d) \in Obj(\mathcal{M}^{d+1})$,
$\delta_{a_0} \cdots \delta_{a_d} F = 0$.

EX: $\mathcal{M} = FI$, $\mathcal{A} = Ab$. F IS STRONGLY POLYNOMIAL AND WITH VALUE IN f.g ABELIAN GPS $\Leftrightarrow$ F IS f.g AS FI-MODULE.

(2)

PROBLEM: THIS NOTION DOES NOT BEHAVE WELL ~~[illegible]~~ RELATIVE TO SUBOBJECTS.

EX: $M = FI$ $\mathbb{Z}$ POLY STRONG DEGREE 0

$\mathcal{A} = Ab$

$\forall n \in \mathbb{N}$, $\mathbb{Z}_{\geq n} : E \longmapsto \begin{cases} \mathbb{Z} & \text{if } |E| \geq n \\ 0 & \text{else} \end{cases}$

$\mathbb{Z}_{=n} : E \longmapsto \begin{cases} \mathbb{Z} & |E| = n \\ 0 & \text{else} \end{cases}$

$$\begin{tikzcd} \mathbb{Z}_{\geq n} \arrow[r, hook] \arrow[d, two heads] & \mathbb{Z} \\ \mathbb{Z}_{=0} & \end{tikzcd}$$

$\delta_n \mathbb{Z}_{\geq n} \simeq \mathbb{Z}_{=n-1}$

$\delta_n \mathbb{Z}_i \simeq \mathbb{Z}_{i-1}$

$\mathbb{Z}_{\geq n}$ AND $\mathbb{Z}_{=n}$ ARE POLYNOMIAL OF STRONG DEGREE n.

WEAK POLYNOMIAL FUNCTORS

DEF: WE DENOTE $SN(\mathcal{M}, \mathcal{A})$ THE FULL SUBCAT OF $Fct(\mathcal{M}, \mathcal{A})$ OF FUNCTORS s.t. $F = \sum_{n \in Ob(\mathcal{M})} k_n(F)$

(EX: IN FI-MODULES, THIS IS EQUIVALENT TO $\operatorname{colim}_{n \in \mathbb{N}} F(\underline{n}) = 0$)

$SN(\mathcal{M}, \mathcal{A})$ is A LOCALIZING SUBCATEGORY OF $Fct(\mathcal{M}, \mathcal{A})$, THICK AND STABLE UNDER COLIMITS.

$\rightarrow \; St(\mathcal{M}, \mathcal{A}) := Fct(\mathcal{M}, \mathcal{A}) / SN(\mathcal{M}, \mathcal{A})$

THE FUNCTORS τ_n STILL INDUCE EXACT ENDOFUNCTORS OF $St(M, A)$. $\longrightarrow$ $\delta_n = \mathrm{coker}(Id \longrightarrow \tau_n)$ IN $St(M, A)$

HAS BECOME MONO

$\Rightarrow$ δ_n IS EXACT.

DEF: (i) AN OBJECT X OF $St(M, A)$ IS SAID POLYNOMIAL OF DEGREE $\leq d$ IF $\forall a_0, \dots a_d$, $\delta_{a_0} \dots \delta_{a_d}(X) = 0$

(ii) A FUNCTOR $F : M \longrightarrow A$ IS WEAKLY POLYNOMIAL IF ITS IMAGE IN $St(M, A)$ IS POLY OF DEG $\leq d$.

NOTE: THE BAD EXAMPLE FROM ABOVE DISAPPEARED.

DENOTE $\mathcal{P}ol_d(M, A)$ THEIR FULL SUBCAT.

FACT: IS A BILOCALIZING SUBCATEGORY OF $St(M, A)$.

WE CAN FORM $\mathcal{P}ol_d(M, A) / \mathcal{P}ol_{d-1}(M, A)$

MAIN RESULT OF (D-V): DESCRIPTION OF WHEN M = CATEGORY OF HERMITIAN OBJECTS (FOR EX $S(A)$)

EX: • $\mathcal{P}ol_0(M, A) \simeq A$ ($\equiv$ CONSTANT FUNCTORS)

• $\mathcal{P}ol_d(S(\mathbb{Z}), Ab) / \mathcal{P}ol_{d-1}$ $\simeq \mathbb{Z}[\mathfrak{S}_d]$-BIMODULES

SYMM GROUP

CHURCH-ELLENBERG-FARB-NAGPAL:

FCT(FI, $\mathcal{A}$) IS LOCALLY NOETHERIAN

└ LOCALLY NOETHERIAN CAT
(i.e. $\exists$ SET OF NOETHERIAN GENERATORS)

CONSEQUENCE: UNDER MILD ASSUMPTIONS ON $\mathcal{M}$ (e.g. $\mathcal{M} = S(A)$) IF $F : \mathcal{M} \to \mathcal{A}$ IS STRONGLY POLYNOMIAL AND TAKES VALUES IN NOETHERIAN OBJECTS OF $\mathcal{A}$, THEN F IS NOETHERIAN.

THM (i) IF A IS A FINITE RING, THEN $\mathcal{P}ol(S(A), Ab)$ IS LOCALLY NOETHERIAN

(ii) $\mathcal{P}ol(S(\mathbb{Z}), Ab)$ IS "ALMOST" LOCALLY NOETHERIAN

└ $\mathbb{Z}[GL_n(\mathbb{Z})]$ NOT LOCALLY NOETHERIAN...

(GENERAL CONSTRUCTION: $\mathcal{M} \rightsquigarrow \tilde{\mathcal{M}}$

0 INITIAL — 0 IS NUL

eg $FI \rightsquigarrow FI_\#$

$$\mathcal{P}ol_d(\mathcal{M}, \mathcal{A}) / \mathcal{P}ol_{d-1} \simeq \mathcal{P}ol_d(\tilde{\mathcal{M}}, \mathcal{A}) / \mathcal{P}ol_{d-1}$$
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