HOMOLOGICAL STABILITY FOR AUTOMORPHISM GROUPS OF FREE PRODUCTS VIA POLYNOMIAL FUNCTORS

(J. WITH COLLINET AND GRIFFIN)

<u>THM</u>: FOR ANY GROUP G WHICH IS INDECOMPOSABLE FOR THE FREE PRODUCT $*$ AND NOT ISOMORPHIC TO $\mathbb{Z}$, THE NATURAL MAP

$$H_i(\mathrm{Aut}(G^{*n});\mathbb{Z}) \longrightarrow H_i(\mathrm{Aut}(G^{*n+1});\mathbb{Z})$$

IS AN ISOMORPHISM IF $n \geq 2i+2$.

(PARTICULAR CASE OF A SLIGHTLY MORE GENERAL THM BY THE SAME AUTHORS)

CONJECTURED BY N. WAHL AND A. HATCHER, WHO PROVED IT FOR $G = \mathbb{Z}/q$ $q = 2,3,4,6$ AND ...

HOMOLOGICAL STABILITY AND FUNCTOR CATEGORIES

(CAN PROBABLY EXTEND TO TWISTED COEFFICIENTS, AND CAN COMPUTE THE COLIMIT [GRIFFIN])

$(\mathcal{C}, *, 0)$ SMALL SYMMETRIC MONOIDAL CAT WITH THE UNIT 0 INITIAL. $A \in \mathrm{Obj}\,\mathcal{C}$

$$\cdots \longrightarrow H_i(\mathrm{Aut}(A^{*n})) \longrightarrow H_i(\mathrm{Aut}(A^{*n+1})) \longrightarrow \cdots$$

<u>QUESTION</u>: DOES IT STABILIZE?

THE USE OF FUNCTOR CATEGORIES CAN HELP TO SHOW TWISTED HOMOLOGICAL STABILITY FROM UNTWISTED STABILITY, <u>BUT</u> WE ALWAYS NEED HIGH ACYCLICITY OF GOOD COMPLEXES WITH ACTION OF $\mathrm{Aut}(A^{*n})$.

ANOTHER RELATED QUESTION : $(\mathcal{D}, +, 0)$ ANOTHER MONOIDAL CATEGORY (SAME ASSUMPTION).
LET US GIVE A MONOIDAL FUNCTOR $\Phi : \mathcal{C} \to \mathcal{D}$.
$\longrightarrow$ CAN BUILD GENERALIZED CONGRUENCE GROUP

$$\Gamma_\Phi(X) := \ker\left(\mathrm{Aut}_{\mathcal{C}}(X) \longrightarrow \mathrm{Aut}_{\mathcal{D}}(\Phi(X))\right)$$

$X \in \mathrm{Obj}\, \mathcal{C}$

$\rightsquigarrow$ HOMOLOGICAL STABILITY FOR $\Gamma_\Phi(A^{*n})$? – THOUGH SHOULD NOT EXPECT STABILITY IN THE USUAL SENSE (GENERALLY) BUT WE CAN LOOK AT $X \mapsto H_i(\Gamma_\Phi(X))$ AS A FUNCTOR ON SOME CATEGORY.
(FOR $H_i(\mathrm{Aut}(A^{*n}))$, HAVE NO ADDITIONAL/FUNCTORIAL STRUCTURE LEFT)
(SUPPOSE THAT Φ_* IS SURJECTIVE)

$$\rightsquigarrow E^2_{p,q} = H_p\left(\mathrm{Aut}_{\mathcal{D}}(\Phi(X)); H_q(\Gamma_{\Phi}(X))\right) \Rightarrow H_{p+q}(\mathrm{Aut}_{\mathcal{C}}(X))$$

USUALLY, BAD IDEA TO USE $H_*(\Gamma_{\Phi}(X))$ TO STUDY $H_*(\mathrm{Aut}_{\mathcal{C}}(X))$ VIA THE SS, BUT IN THE PARTICULAR CASE OF $H_*(\mathrm{Aut}(G^{*n}))$, THERE IS A NICE "CONGRUENCE SUBGROUP" (GROUP OF FOUXE-RABINOVICH) WHICH CAN BE USED.

RECOLLECTIONS ON $\mathrm{Aut}(G_1 * \dots * G_n)$

FAMILY $G = (G_i)_{i \in E}$, $\underline{G} = \mathop{*}_{i \in E} G_i$. WHAT IS $\mathrm{Aut}(\underline{G})$?
HAVE (1) GROUP ~~[illegible]~~ MONOMORPHISM $\mathrm{aut}(G) := \prod_{i \in E} \mathrm{Aut}(G_i) \hookrightarrow \mathrm{Aut}(\underline{G})$
(2) ISOMORPHIC FACTORS CAN BE PERMUTED: SUPPOSE ON ISO IS CHOSEN, IF S = SET OF EQUIVALENCE

CLASSES OF E FOR $i \sim j \Leftrightarrow G_i \simeq G_j$

$\rightsquigarrow \Sigma := \prod_{s \in S} \mathfrak{S}_s \hookrightarrow \mathrm{Aut}(\underline{G})$

PERMUTATIONS OF THE ELEMENT IN THE EQUIVALENCE CLASS.

(3) PARTIAL CONJUGATIONS: FOR $i \neq j \in E$, WE HAVE A GROUP MONOMORPHISM $\alpha_{ij} : G_i \longrightarrow \mathrm{Aut}(\underline{G})$

FOR $x \in G_i$, $\alpha_{ij}(x) : g \in G_t \longmapsto \begin{cases} g & \text{if } t \neq j \\ xgx^{-1} & \text{if } t = j \end{cases}$

DEF: THE FOUXE-RABINOVICH GROUP $FR(G)$ OF G IS THE SUBGROUP OF $\mathrm{Aut}(\underline{G})$ GENERATED BY ALL THE IMAGES OF THE α_{ij}'s.

($\leftrightarrow$ ELEMENTARY MATRICES)

WE CAN WRITE A GENERAL PRESENTATION OF $FR(G)$ (INDEPENDENT OF G).

WE HAVE IN FACT A GROUP MONOMORPHISM

$$(\Sigma \ltimes \mathrm{aut}(\underline{G})) \ltimes FR(\underline{G}) \hookrightarrow \mathrm{Aut}(\underline{G})$$

$=: \Sigma\mathrm{Aut}(\underline{G})$ = SYMMETRIC AUTOMORPHISM GROUP.

CONSEQUENCE OF KUROSH THEOREM: IF THE GROUPS G_i ARE INDECOMPOSABLE FOR $*$ AND NOT ISOMORPHIC TO $\mathbb{Z}$, THEN $\Sigma\mathrm{Aut}(\underline{G}) = \mathrm{Aut}(\underline{G})$.

(THE MORE GENERAL THEOREM IS FOR ANY GROUP BUT ~~[illegible]~~ REPLACING THE AUTOMORPHISM GROUP BY THE SYMMETRIC AUTOMORPHISMS.)

FUNCTORIALITY

LET Γ BE THE CATEGORY OF FINITE SETS WITH PARTIAL FUNCTIONS. LET $\mathcal{C}$ BE ANY CATEGORY.

WE DEFINE THE WREATH PRODUCT $\Gamma \int \mathcal{C}$ AS FOLLOWS:

i) OBJECTS: FINITE FAMILIES OF OBJECTS OF $\mathcal{C}$: $(A, (C_a)_{a\in A})$
$A \in \mathrm{Obj}\,\Gamma$, $C_a \in \mathrm{Obj}\,\mathcal{C}$

ii) MORPHISMS $(A, (C_a)_{a\in A}) \to (A', (C'_{a'})_{a'\in A'})$ ARE PAIRS $(u, (\varphi_a)_{a\in \mathrm{def}(u)})$ WHERE $u : A \to A' \in \mathrm{Mor}\,\Gamma$ DEFINED ON $B = \mathrm{def}(u) \subset A$ AND $\varphi_a : C_a \to C'_{u(a)}$ MORPHISM OF Γ.

FOR Γ' A SUBCATEGORY OF Γ, WE HAVE A SUBCATEGORY $\Gamma' \int \mathcal{C}$.

NICE SUBCATEGORIES OF Γ:

- Σ CATEGORY OF BIJECTIONS
- Θ CATEGORY OF EVERYWHERE DEFINED INJECTIONS
  "FI"
- $\tilde{\Theta}$ CATEGORY OF PARTIALLY DEFINED INJECTIONS
  "FI#"

$$\mathrm{Aut}_{\Gamma\int\mathcal{C}}(\underbrace{A, \ldots, A}_{n}) = S_n \int \mathrm{Aut}(A)$$

IF $(\mathcal{C}, *, 0)$ IS A SYM. MONOIDAL CATEGORY (RESP. WITH 0 INITIAL OR 0 NUL), WE HAVE A FUNCTOR

- $\Sigma \int \mathcal{C} \longrightarrow \mathcal{C} : (C_a)_{a\in A} \longmapsto \mathop{*}\limits_{a\in A} C_a$

(0 INITIAL) • $\Theta \int \mathcal{C} \longrightarrow \mathcal{C}$

(0 NUL) • $\tilde{\Theta} \int \mathcal{C} \longrightarrow \mathcal{C}$

FR DEFINES A FUNCTOR $\tilde{\Theta}\int \text{Groups} \to \text{Groups}$

$$(G_1, \ldots, G_n) \longmapsto FR(G_1, \ldots, G_n)$$

ON MORPHISMS: $\underset{G_i}{\alpha_{ij}(\underset{\cap}{x})} \longmapsto \begin{cases} \alpha_{u(i),u(j)}(\varphi_i(x)) & \text{IF } u(i) \text{ AND } u(j) \text{ ARE DEFINED} \\ 1 & \text{ELSE} \end{cases}$

MORPH IN $\tilde{\Theta}\int\text{Groups}$: $(G_i)_{i\in E} \longrightarrow (G'_a)_{a\in E'}$

$$u : E \to E'$$
$$\varphi_i : G_i \longrightarrow G'_{u(i)}$$

(THE USUAL PRESENTATION OF $FR(G)$ IMPLIES IT IS FUNCTORIAL)

$$\Sigma \operatorname{Aut}(G) = \underbrace{\left(\Sigma \ltimes \operatorname{aut}(G)\right)}_{\operatorname{Aut}_{\tilde{\Theta}\int\text{Groups}}(G)} \ltimes FR(G)$$

<u>POLYNOMIAL FUNCTORS</u>

WE HAVE A CROSS-EFFECT FUNCTOR

$$\underset{\substack{\text{ANY}\\\text{USUAL}\\\text{CATEGORY}}}{Fct(\tilde{\Theta}\int\mathcal{C}}, \underset{\substack{\text{ABELIAN}\\\text{GROUPS}}}{Ab}) \xrightarrow{cr} Fct(\Sigma\int\mathcal{C}, Ab)$$

(EQUIVALENCE OF CATEGORIES [PIRASHVILI])

S.t. FOR ANY $F : \tilde{\Theta}\int\mathcal{C} \longrightarrow Ab$ AND ANY OBJECT

$c = (C_a)_{a\in E}$, $F(c) \simeq \bigoplus_{A\subset E} cr(F)(C|_A)$

<u>DEF</u>: $F : \tilde{\Theta}\int\mathcal{C} \to Ab$ is POLYNOMIAL OF DEGREE $\leq d$ IF $cr(F)(C) = 0$ WHEN $\#C > d$.

THM: $\forall i \in \mathbb{N}$, THE FUNCTOR

$$\tilde{\Theta} \int \text{GROUPS} \xrightarrow{FR} \text{GROUPS} \xrightarrow{H_i} Ab$$

IS POLYNOMIAL OF DEGREE $\leq 2i$.

THE PROOF INVOLVES CONTRACTIBILITY OF SOME NICE COMPLEXES ... AND ALL OF JAMES GRIFFIN'S THESIS ...

RELATION WITH HOMOLOGICAL STABILITY FOR $H_*(Aut(G^{*n}))$

THM (NAKAOKA) FOR ANY ABELIAN GROUP M (WITH TRIVIAL SYM. GROUP ACTION),

$$H_i(S_n, M) \longrightarrow H_i(S_{n+1}, M)$$

IS AN ISO FOR $n > 2i$.

CONSEQUENCE: (BETLEY): IF $F: \tilde{\Theta} \longrightarrow Ab$ IS POLYNOMIAL OF DEGREE $\leq d$, THEN

$$H_i(S_n, F(n)) \to H_i(S_{n+1}, F(n+1))$$ IS AN ISO FOR $n > 2i+d$.

SKETCH PROOF OF STABILITY FOR $Aut(G^{*n})$ FROM THESE 2 THM'S

(1) $\forall q \in \mathbb{N}$, THE FUNCTOR

$$\tilde{\Theta} \longrightarrow Ab$$
$$E \longmapsto H_q(Aut(G)^E \ltimes FR_E(G))$$

IS POLYNOMIAL OF DEGREE $\leq 2q$

[IDEA: USE HOCHSCHILD-SERRE SS AND PROVE THAT THE E^2 PAGE IS ALREADY POLYNOMIAL OF THAT

~~DEGREE~~ $\cancel{2}$ USING THE FIRST THM.)

(2) APPLY THE RESULT OF ~~BETLEY~~ TO GET THAT

$$E^2_{p,q}(n) \to E^2_{p,q}(n+1) \quad \text{is an ISO for} \quad n > 2(p+q).$$

(SERRE SS FOR $\Sigma(E) \ltimes \left(\mathrm{Aut}(G)^E \ltimes FR_E(G)\right)$