LECTURE V (TOM CHURCH)

DEF: • $FI_{\mathbb{Z}} = S(\mathbb{Z})$

OBJ: FREE f.g. $\mathbb{Z}$-Mod A ($\simeq \mathbb{Z}^n$)

MOR: $A \to B$ ($f: A \hookrightarrow B$, $C \subseteq B$) s.t. $B = f(A) \oplus C$

• $GI_{\mathbb{Z}}$ OBJ = SAME

MOR $A \to B$: ($f: A \hookrightarrow B$, $v_1, \dots, v_\ell \in B$) s.t. $B = f(A) \oplus \langle v_1 \rangle \oplus \dots \oplus \langle v_\ell \rangle$

$$\begin{array}{ccc} GI_{\mathbb{Z}} & & A \\ \downarrow \varepsilon & & \downarrow \\ \mathbb{N}^{=} & & \mathrm{rank}(A) \end{array}$$

• $\mathbb{N}^{=}$ OBJECT $n \in \mathbb{N}$

MORPH $n \to N = \begin{cases} \mathbb{Z}(id) & n = N \\ 0 & n \neq N \end{cases}$

$\varepsilon_! : GI_{\mathbb{Z}}\text{-Mod} \longrightarrow \mathbb{N}^{=}\text{-Mod}$ = SEQUENCE OF ABELIAN GP = GRADED ABELIAN GP.

$H_i^{\varepsilon}(V) = \mathbb{L}_i(\varepsilon)(V) \in \mathbb{N}^{=}\text{-Mod}$

DEF: GIVEN $M \in \mathbb{N}^{=}\text{-Mod}$, $\deg M = \sup\{n \mid M_n \neq 0\}$

WRITE $M \simeq 0$ IF $\deg M < \infty$

THM E (CHORNEY): $H_i^{\varepsilon}(\mathbb{Z}) \simeq 0 \quad \forall i$

WHAT ARE CONSEQUENCES OF $H_i^{\varepsilon}(\mathbb{Z}) \simeq 0$?

FACTOR

$$\begin{array}{ccc} A & GI_{\mathbb{Z}} & \\ \downarrow & \downarrow \pi & \searrow \varepsilon \\ \mathrm{rank}\, A & \mathbb{N}^{\geq} & \xrightarrow{\varepsilon'} \mathbb{N}^{=} \end{array}$$

$\mathbb{N}^{\geq 0} = \begin{cases} \text{Obj} = \mathbb{N} \\ n \to m = \begin{cases} * & n \leq N \\ \emptyset & n > N \end{cases} \end{cases}$

$\mathcal{E} = e \circ \pi \Rightarrow \mathcal{E}_! = e_! \circ \pi_!$

$\rightsquigarrow$ HAVE A SS $\underbrace{H_p^e(H_q^\pi(V))}_{E^2_{p,q} =} \Rightarrow H^{\mathcal{E}}_{p+q}(V)$

<u>STEP 1</u>:

$$\begin{tikzcd} H_q^\pi(V) = & n \arrow[r, mapsto] \arrow[d] & H_q(Gl_n\mathbb{Z}, V_n) \arrow[d] \\ & N & H_q(Gl_N\mathbb{Z}, V_N) \end{tikzcd}$$

<u>1a</u>: GIVEN A $GI_\mathbb{Z}$-MODULE V,

$$(\pi_!(V))_n = (V_n)_{Gl_n(\mathbb{Z})}$$

(GIVEN M A $\mathbb{N}_\geq$ MODULE, $M_0 \to M_1 \to M_2 \to \cdots$)
($(\pi^* M)_n = M_n$ SEEN AS A TRIVIAL $Gl_n\mathbb{Z}$-MODULE)

HOW WOULD WE COMPUTE $H_q^\pi(V)$? — RESOLVE V BY FREE $GI_\mathbb{Z}$-MODULES AND APPLY $\pi_!$.

<u>DEF</u>: $M(d) = \mathbb{Z}\mathrm{Hom}_{FI_\mathbb{Z}}(\mathbb{Z}^d, -)$ i.e. BY SUCH

$F(d) = \mathbb{Z}\mathrm{Hom}_{GI_\mathbb{Z}}(\mathbb{Z}^d, -)$

$F(0)$: $\mathbb{Z} \to \mathbb{Z}Gl_1\mathbb{Z} \to \mathbb{Z}Gl_2\mathbb{Z} \to \mathbb{Z}Gl_3\mathbb{Z} \to \cdots$

$F(1)$: $0 \to \mathbb{Z}Gl_1\mathbb{Z} \to \mathbb{Z}Gl_2\mathbb{Z} \to \mathbb{Z}Gl_3\mathbb{Z} \to \cdots$

$F(2)$: $0 \to 0 \to \mathbb{Z}Gl_2\mathbb{Z} \to \mathbb{Z}Gl_3\mathbb{Z} \to \cdots$

⇝ RESOLUTIONS OF V LOOK LIKE

$$
\begin{tikzcd}
\vdots \arrow[d] & & & & & \\
F(3) \arrow[d] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}Gl_3 \arrow[r] & \mathbb{Z}Gl_4 \arrow[r] & \cdots \\
F(1) \arrow[d] & 0 \arrow[r] & \mathbb{Z}Gl_1 \arrow[r] & \mathbb{Z}Gl_2 \arrow[r] & \mathbb{Z}Gl_3 \arrow[r] & \mathbb{Z}Gl_4 \arrow[r] & \cdots \\
V & V_0 \arrow[r] & V_1 \arrow[r] & V_2 \arrow[r] & V_3 \arrow[r] & V_4 \arrow[r] & \cdots
\end{tikzcd}
$$

→ EACH COLUMN IS A FREE RESOLUTION OF V_i BY FREE $Gl_n\mathbb{Z}$-MODULES. → JUST COMPUTING GROUP HOMOLOGY IN EACH COLUMN. (ALWAYS POSSIBLE TO HAVE SUCH RESOLUTIONS BECAUSE THE $F(d)$'S ARE PROJECTIVE GENERATORS)

$$\left(\left(\varepsilon_!(V)\right)_n = (V_n)_{Gl_n\mathbb{Z}} \Big/ \mathrm{Im}(V_{n-1})/Gl_{n-1}\mathbb{Z} = (V_n)_{Gl_n\mathbb{Z}} \Big/ \mathrm{Im}\left((V_{n-1})_{Gl_n\mathbb{Z}}\right)\right.$$

$$= \left(V_n / V_{n-1}\right)_{Gl_n\mathbb{Z}}$$

~~[illegible]~~ Rem: $\mathbb{N}^{\geq}$-mod $M \longleftrightarrow M_0 \xrightarrow{T} M_1 \xrightarrow{T} M_2 \xrightarrow{T} \cdots$

GRADED $\mathbb{Z}[T]$-MODULE

$$H_q^{\pi}(V) : \cdots \to H_q(Gl_n\mathbb{Z}, V_n) \xrightarrow{T} H_q(Gl_{n+1}\mathbb{Z}, V_{n+1}) \to \cdots$$

STEP 2: $H_p^e(M) = ?$

$$e_!(M) = M/TM = H_0^e(M), \quad H_1^e(M) = \ker(M \xrightarrow{T} M)$$

$$H_i^e(M) = 0 \qquad i \geq 2.$$

$\underline{SS}$: $E^2_{pq} = H^e_p(H^\pi_q(\underline{\mathbb{Z}}))$

$E^2_{0q} = n \longmapsto \operatorname{coker}(H_q(Gl_{n-1}\mathbb{Z}) \to H_q(Gl_n\mathbb{Z}))$

$E^2_{1q} : n \longmapsto \ker(H_q(Gl_n\mathbb{Z})) \to H_q(Gl_{n+1}\mathbb{Z}))$

$$
E^2_{pq}\quad
\begin{array}{|cccccc}
\vdots & \vdots & \vdots & \vdots & & \\
\operatorname{coker} & \ker & 0 & 0 & \cdots & 0 \\
\operatorname{coker} & \ker & 0 & 0 & \cdots & \\
\vdots & \vdots & \vdots & \vdots & & \vdots
\end{array}
\quad \Longrightarrow H^{\varepsilon}_{p,q}(\underline{\mathbb{Z}}) \simeq 0
$$

$\longrightarrow$ NO POSSIBLE DIFFERENTIAL!

$\longrightarrow H_i(Gl_n\mathbb{Z}) \longrightarrow H_i(Gl_{n+1}\mathbb{Z})$ ISO FOR $n \gg 0$.

$\tilde{\Gamma}$: Obj = PAIRS (FREE f.g $\mathbb{Z}$-MODULE A, $b_A \subset A/pA$ BASIS)

$\operatorname{Mor}((A, b_A) \to (B, b_B))$ ($f: A \hookrightarrow B$, $\{v_1 \dots v_\ell\} \subset B$ s.t. $B = f(A) \oplus \langle v_1 \rangle \oplus \dots \oplus \langle v_\ell \rangle$, $b_B = \bar{f}(b_A) \cup \{\bar{v}_1, \dots, \bar{v}_\ell\}$)

$\mathcal{S}$ = FINITE SETS, $\mathbb{Z}(\text{Bij})$

$$
\begin{tikzcd}
\tilde{\Gamma} \arrow[d, "\pi"'] \arrow[dr, "\varepsilon_\Gamma"] & \\
FI \arrow[r, "e"'] & \mathcal{S}
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{\Gamma}\text{-Mod} \arrow[d] \arrow[dr] & \\
FI\text{-Mod} \arrow[r] & \mathcal{S}\text{-Mod}
\end{tikzcd}
\ \text{, } \exists q
$$

$\Gamma_n(p)$ = PREIMAGE OF $\{Id\} \subset GL_n\mathbb{F}_p$

$\tilde{\Gamma}_n(p)$ = PREIMAGE OF $\{S_n\} \subset GL_n\mathbb{F}_p$

$\underline{\text{STEP 1}}$: $H^\pi_q(\underline{\mathbb{Z}}) = H_q(\Gamma_n(p))$

$\underline{\text{STEP 2}}$: $H^e_p(V) = H^{FI}_p(V)$

$\rightsquigarrow$ SS $\quad H_0^{FI}(\mathcal{H}_q) \quad H_1^{FI}(\mathcal{H}_q) \quad H_2^{FI}(\mathcal{H}_q) \; \cdots$

$\qquad\qquad\qquad\qquad\qquad\qquad H_2^{FI}(\mathcal{H}_{q-1}) \; \cdots$

GIVES $\rightsquigarrow$ ~~ASSERTION~~: $H_p^{FI}(\mathcal{H}_k) \approx 0 \quad k<q \overset{SS}{\Longrightarrow} H_0^{FI}(\mathcal{H}_q) \approx 0$, $H_1^{FI}(\mathcal{H}_q) \approx$ $\overset{\text{THM B}}{\Longrightarrow} H_p^{FI}(\mathcal{H}_q) \approx 0$

GIVES THE EXPONENTIAL BOUND!

REM: THE ARGUMENT WORKS FOR VERY GENERAL RINGS (SAYS ANDY) AND THUS ONE SHOULDN'T EXPECT TOO MUCH...

GOAL: $H_i^{\varepsilon}(\underline{\mathbb{Z}}) \approx 0$

DEF: $B_\ell : GI_2\text{-Mod} \longrightarrow GI_2\text{-Mod}$

$$B_\ell V \qquad A \longmapsto \bigoplus_{\substack{C \subseteq A \\ v_1, \dots, v_\ell \\ A = C \oplus \langle v_1 \rangle \oplus \cdots \oplus \langle v_\ell \rangle}} V_C$$

LEMMA: $H_i^{\varepsilon}(B_\ell(V)) = H_i^{\varepsilon}(V)[\ell]$

OBSERVATION: $B_\ell(\underline{\mathbb{Z}}) = M(\cancel{A})$

LEMMA: $H_i^{\varepsilon}(\underline{\mathbb{Z}}) \approx 0$ FOR $i \leq I$

$\Rightarrow H_i^{\varepsilon}(M(A)) \approx$ FOR $i \leq I$

Recall: COMPUTED $H_i^{FI}(\mathbb{Z}_0) = \varepsilon_i$

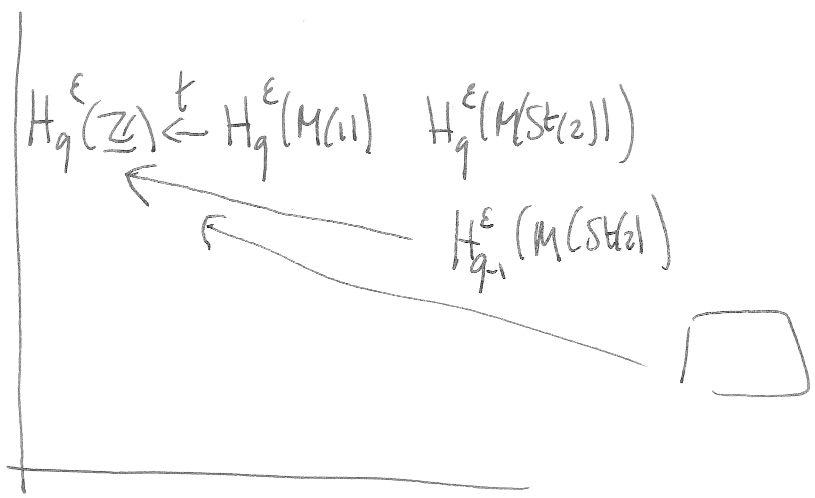
+ NOTED $H_i^{FI_{\mathbb{Z}}}(\mathbb{Z}_0) = St_i$

$$\cdots \longrightarrow M(St_2) \longrightarrow M(1) \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathbb{Z}_0$$

RESOLUTION BY $FI_{\mathbb{Z}}$-MODULE. CAN THINK OF IT AS A RESOLUTION BY (NON-FREE!) $GL_{\mathbb{Z}}$-MODULE

$\rightsquigarrow$ SS $\quad E^1_{pq} = H_q^{\varepsilon}(M(St_p)) \Rightarrow H^{\varepsilon}_{p+q}(\mathbb{Z}_0) \simeq 0$

$$\begin{array}{llll} 0 & & & \\ \downarrow & & & \\ F(1) & 0 \to GL_1\mathbb{Z} \to \cdots & & \\ \downarrow & & & \\ F(0) & \mathbb{Z} \to GL_1\mathbb{Z} \to \cdots & & \\ \downarrow & & & \\ \mathbb{Z}_0 & \mathbb{Z} \to 0 \to 0 \cdots & & \\ \downarrow & & & \\ 0 & & & \end{array}$$



IF V ~~is an~~ comes from an $FI_{\mathbb{Z}}$-MOD, $\exists\ B_1 V \xrightarrow{t} V$

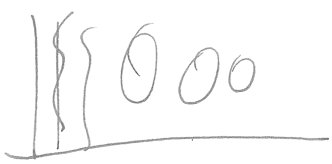
eg $\quad B_1\underline{\mathbb{Z}} \xrightarrow{t} \underline{\mathbb{Z}}$
$\quad \| $
$\quad M(1) \xrightarrow{t} \underline{\mathbb{Z}}$

$\longleftarrow$ THIS IS THE d_1-DIFF IN THE SS

$$t_*: H_q^{\varepsilon}(M(1)) \longrightarrow H_q^{\varepsilon}(\mathbb{Z})$$
$$\|$$
$$H_q^{\varepsilon}(\mathbb{Z})[1]$$

Claim: $t^2 = 0$. $\quad t_*: H_q^{T}(\underline{\mathbb{Z}})[1] \longrightarrow H_q^{T}(\mathbb{Z})$

$t_* = T = 0$ ON E^2_{pq}



$\Rightarrow t_*$ ACTS BY 0 ON E^{∞}

$\Rightarrow$ ACTS BY 0 ON FILTRATION QUOTIENTS

BUT ONLY 2 TERMS IN THE FILTRATIONS $\Rightarrow t_*^2 = 0$