LECTURE IV (TOM CHURCH)

PROOF OF THM A : FOR $n \geq 9 \cdot 2^i - 3$

(PUTMAN, C-E)

$$H_i(\Gamma_n(p)) \simeq \frac{\bigoplus_{j=1}^{n} H_i(\Gamma_{n-1}^{(j)}(p))}{\operatorname{Im} \bigoplus_{j,k=1}^{n} H_i(\Gamma_{n-2}^{(j,k)}(p))}$$

TWO INGREDIENTS:

THM B (C-E): FOR ANY FI-MODULE V, $\deg H_k^{FI}(V) \leq \deg H_0 + \deg H_1 + k - 1$

THM C (C-E): THM A IS EQUIVALENT TO

$$\left.\begin{array}{l} \deg H_0^{FI}(H_q) \\ \deg H_1^{FI}(H_q) \end{array}\right\} \leq 9 \cdot 2^q - 3$$

$\to$ idea: BECAUSE $\exists$ A CPX OF FI-MODULES COMPUTING $H_*^{FI}(V)$ (FOR ANY FI-MODULE V)

$$\cdots \longrightarrow \bigoplus_{i \neq j = 1}^{n} V_{[n]-i-j} \longrightarrow \bigoplus_{i=1}^{n} V_{[n]-i} \to V_{[n]} \to 0$$

$$\rightsquigarrow H_*^{FI}(V)_n$$

PUTMAN: CENTRAL STABILITY $\underset{\text{ROUGHLY}}{\longleftrightarrow}$ CONDITION OF THM A

(THE ABOVE COMPLEX IS IN PUTMAN'S PAPER (LEM. 4.4)
AND PROP 4.5 $\longleftrightarrow$ THM B OVER A FIELD OF LARGE CHARACT.)
— ALSO IN CEFN, DEF 2.7.

RECALL: $H_0^{FI}(V) = K \underset{FI}{\otimes} V$ WHERE K IS THE FI-BIMODULE

$$K(S,T) = \begin{cases} \mathbb{Z}\mathrm{Hom}(S,T) & \text{IF } S \simeq T \\ 0 & \text{IF } S \not\simeq T \end{cases}$$

ALSO, DEFINE $\mathcal{S}$ WITH OBJECTS FINITE SETS AND $\mathrm{Hom}_{\mathcal{S}}(S,T) = \begin{cases} \mathbb{Z}\,\mathrm{Inj}(S,T) & S \simeq T \\ 0 & S \not\simeq T \end{cases}$

PREVIOUSLY, COMPUTED $H_i^{FI}(V)$ BY RESOLVING V

"$\mathrm{Tor}_i^{FI}(k,V)$"

INSTEAD, RESOLVE k BY $C_\bullet \longrightarrow k$

THEN $C_\bullet \otimes V$ WILL BE THE CPX MENTIONED ABOVE.

<u>COMPARE</u>: $k[T]$-MODULE $M \rightsquigarrow H_0^{k[T]}(M) = M/TM = k \otimes_{k[T]} M$

AND $H_i^{k[T]}(M) = \mathrm{Tor}_i^{k[T]}(k, M)$

k HAS RESOLUTION

$$0 \to k[T] \xrightarrow{\cdot T} k[T] \to k$$

$$\rightsquigarrow \left(k[T] \xrightarrow{T} k[T]\right) \otimes M = M \xrightarrow{T} M$$

$$\to H_0(M) = M/TM, \quad H_1(M) = \ker(T : M \to M),$$

$$H_i(M) = 0 \quad i > 1.$$

<u>RECALL</u>: FOR A A GRADED ALGEBRA, $A_0 \simeq k$, $A_+ = A_{\geq 1}$

~~~~ BAR RESOLUTION $B(A_+, A_+, A) \to A$

$$\cdots \to A_+ \otimes_k A_+ \otimes_k A \longrightarrow A_+ \otimes_k A \longrightarrow A$$

$$\rightsquigarrow B(A_+, A_+, A) \otimes M = B(A_+, A_+, M)$$

DEFINE A = FI-BIMODULE $A(S,T) = \mathbb{Z}\mathrm{Hom}_{FI}(S,T)$

A_+ = FI-BIMODULE $A_+(S,T) = \begin{cases} 0 & \text{if } S \simeq T \\ \mathbb{Z}\mathrm{Hom}_{FI}(S,T) & \text{if } S \neq T \end{cases}$

$\tilde{L}_0 = A$

$\tilde{L}_1 = A_+ \otimes_S A$

$\tilde{L}_2 = A_+ \otimes_S A_+ \otimes_S A$ $\rightsquigarrow \cdots \to \tilde{L}_3 \to \tilde{L}_2 \to \tilde{L}_1 \to \tilde{L}_0$

$\vdots$

coker = isos OF A
~~~~

L_i IS THE FI-BIMODULE $\qquad \tilde{L}_i \otimes_{FI} V = \underbrace{A_{+S} \otimes_S \cdots \otimes_S A_{+S}}_{L_i} \otimes_S V$

$L_1(S,T)$ HAS BASIS $\left\{ \begin{array}{c} f : S \hookrightarrow T \\ f(S) \subsetneq T \end{array} \right\}$

$L_2(S,T)$ —— $\{ f : S \hookrightarrow T,\ U_1,\ f(S) \subsetneq U_1 \subsetneq T \}$

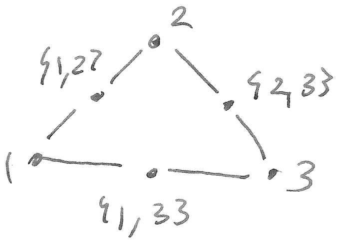
$L_3(S,T)$ —— $\{ f : S \hookrightarrow T,\ U_1, U_2,\ f(S) \subsetneq U_1 \subsetneq U_2 \subsetneq T \}$

$$\rightsquigarrow \quad \bigoplus_{S \subsetneq T \subsetneq [n]} V_S \longrightarrow \bigoplus_{S \subsetneq [n]} V_S \longrightarrow V_n$$

WANT TO REPLACE THIS BY A SMALLER COMPLEX BY REPLACING $L_\bullet$ BY $C_\bullet$

$L_\bullet(S,T)$ IS $\bigoplus_f$ THE REDUCED CHAINS OF THE POSET OF PROPER NON-EMPTY SUBSETS OF $T - f(S)$.

EX: $|T| = |S| + 3$



$L_\bullet(S,T) \simeq \partial \Delta^{|T|-|S|-1}$ —— CAN BE REPLACED BY A MUCH SMALLER THING:

$$C_i(S,T) = \begin{cases} 0 & \text{if } |T| \neq |S| + i \\ \{f : S \hookrightarrow T\} \boxtimes \mathbb{Z}_{\varepsilon_{T-f(S)}} & \text{if } |T| = |S| + i \end{cases}$$

$\leftarrow$ SIGN REPRESENTATION OF $T - f(S)$

$$\rightsquigarrow \quad (C_i \otimes_S V)_T = \bigoplus_{\substack{S \subseteq T \\ |S| = |T| - i}} V_S \boxtimes \varepsilon_{T-S}$$

CAN REPLACE FI BY $S(\mathbb{Z})$ OR $S(\mathbb{F}_p)$
BUT THE POSET IS NOT $\partial\Delta$ BUT CHARNEY'S "SPLIT BUILDING", WHICH IS COHEN-MACCAULEY

$$\rightsquigarrow \quad \cdots \longrightarrow \bigoplus_{\substack{C \subseteq \mathbb{F}_p^n \\ \dim C = n-2}} V_{n-2} \otimes \underbrace{St'(\mathbb{F}_p^2)}_{\text{SPLIT STEINBERG MODULE}} \longrightarrow \bigoplus_{\substack{B \leq \mathbb{F}_p^n \\ \dim B = n-1}} V_{n-1} \longrightarrow V_n$$

(Rem : $H_i^{FI}(\mathbb{Z}_0) \simeq \mathcal{E}_i$)

Def : $GI_{\mathbb{Z}} = \begin{cases} \text{Obj} = \text{FREE FINITE RANK } G\text{-MODULES} \\ \text{Mor}(A,B) = \{ f : A \hookrightarrow B,\ v_1, .., v_\ell \in B \text{ s.t. } B \simeq f(A) \oplus \langle v_1 \rangle \oplus \cdots \oplus \langle v_\ell \rangle \} \end{cases}$

Def: $\tilde{\Gamma} = \begin{cases} \text{OBJ} = \text{FREE f.g } \mathbb{Z}\text{-MODULE } A \text{ WITH A BASIS } b_A \text{ FOR } A/p \\ \text{Mor}((A, b_A),(B, b_B)) = \{ f : A \hookrightarrow B,\ \{v_1, \dots, v_\ell\} \subset B \text{ s.t. } B = f(A) \oplus \langle v_1 \rangle \oplus \cdots \oplus \langle v_\ell \rangle,\ b_B = \bar{f}(b_A) \cup \{\bar{v}_1, \dots, \bar{v}_\ell\} \} \end{cases}$

Def: $\mathbb{N}^= = \begin{cases} \text{obj } n \in \mathbb{N} \\ \text{Mor}(n,m) = \begin{cases} \mathbb{Z} & n = m \\ 0 & n \neq m \end{cases} \end{cases}$

$\varepsilon : GI_{\mathbb{Z}} \longrightarrow \mathbb{N}^=$, $\quad A \longmapsto \operatorname{rank} A$

$\varepsilon_p : \tilde{\Gamma} \longrightarrow \mathcal{B}$, $\quad (A, b_A) \longmapsto b_A$

$\varepsilon_! : GI_{\mathbb{Z}}\text{-Mod} \longrightarrow \mathbb{N}^=\text{-Mod}$, $\quad \varepsilon_{p!} : \tilde{\Gamma}\text{-Mod} \longrightarrow \mathcal{B}\text{-Mod}$

$H_i^{\mathcal{E}}(V)$ DERIVED FUNCTORS $\mathbb{L}_i(\mathcal{E})(V)$.

<u>Claim D</u>: D1: HOMOLOGICAL STABILITY FOR $H_i(GL_n\mathbb{Z};\mathbb{Z})$
$n \geq N(i)$ $\forall i$

$$\Updownarrow$$

$$H_i^{\mathcal{E}}(\underline{\mathbb{Z}})_n = 0 \quad \text{FOR } n \gg 0 \quad \forall i$$
$n \geq N$

CONSTANT FUNCTOR

<u>D2</u>: TWISTED STABILITY FOR $H_i(GL_n\mathbb{Z};V_n)$ $\forall i$

$$\Updownarrow$$

$$H_i^{\mathcal{E}}(V)_n = 0 \quad \text{FOR} \quad n \gg 0$$

<u>D3</u>: THM A FOR $H_i(\Gamma_n(p))$ FOR $n \gg 0$

$$\Updownarrow$$

$$H_i^{\mathcal{E}_\Gamma}(\underline{\mathbb{Z}}) \simeq 0$$