LECTURE III (TOM CHURCH)

FIRST GOAL: PROVE THM (C-E) THAT

$$\deg H_i^{FI}(W) \leq \deg H_0^{FI}(W) + \deg H_1^{FI}(W) + i - 1.$$

PRECURSORS: EISENBUD-FLØYSTAD-WEYMAN (OVER $\mathbb{C}$)
PUTMAN (char $\gg 0$)

REM: BOUND NOT PRESERVED BY "SHIFTING" THE FI-MODULE, WHERE "SHIFT" MEANS

$$0 \to V \to \underset{\text{FREE}}{M} \to W \to 0$$

$$H_i(V) \simeq H_{i+1}(W)$$

BUT THE THM ADDS 2 TO THE BOUND.

$$0 \to V \to M \to W \to 0$$

$V \subseteq M(d)$ is INSIDE A FREE MODULE
$\rightsquigarrow$ V is TORSION-FREE, i.e. $\forall f:[n] \hookrightarrow [N]$,
AND $\forall v \in V_n$, $f_*(v) = 0 \Rightarrow v = 0$.

BECAUSE $M(d)$ is TORSION FREE.

IDEA: SHIFTING W TWICE "DOWN" WILL MAKE IT NICE:

$$0 \to U \to \underset{\text{FREE}}{M'} \to V \to 0$$

WHAT DOES U HAVE THAT V DOESN'T?
ANSWER: U is "SATURATED"

$V \subseteq M(1)$ IS NOT SATURATED: $2e_1 \in V_3$ BUT $2e_1 \notin V_1$
(FROM THE EXAMPLE)

DEF: $\mathcal{U} \subset M(A)$ IS SATURATED IF FOR ANY $m \in M_n$, ANY $f : [n] \hookrightarrow [N]$, $f_*(m) \in \mathcal{U}_N \Rightarrow m \in \mathcal{U}_n$.

INTRINSIC VERSION: A TORSION-FREE FI-MODULE $\mathcal{U}$ IS SATURATED IF $\forall u \in \mathcal{U}_T$, T SET s.t. $[f, f' : T \hookrightarrow R$ WITH $f|_S = f'|_S] \Longrightarrow [f_*(u) = f'_*(u)]$
THEN $u \in \mathrm{im}(\mathcal{U}_S \to \mathcal{U}_T)$.

EQUIVALENTLY, $\forall k$, $H_1(S_{+k} V)_2 = 0$

k-SHIFT OF V

$S_{+1} V = V \circ (\{\ast\} \sqcup -) : FI \to FI \to \mathbb{Z}\text{-Mod}$

PROP: $0 \to \mathcal{U} \to \underset{\text{FREE}}{M} \to \underset{\text{TORSION FREE}}{V} \to 0 \Longrightarrow \mathcal{U}$ SATURATED.

PF: IF $m \in M_n$ AND $f : n \hookrightarrow N$ s.t. $f_*(m) \in \mathcal{U}_N$

$\bar{m} \in V_n \quad f_*(\bar{m}) = \overline{f_*(m)} = 0 \Rightarrow \bar{m} = 0$
$\Rightarrow m \in \mathcal{U}_n$

REM: SATURATEDNESS OF $\mathcal{U}$ SAYS PRECISELY THAT YOU CAN RECOVER THE FI-MODULE $\mathcal{U}$ FROM
$\mathrm{colim}\, \mathcal{U} = \bigcup_{n \in \mathbb{N}} \mathcal{U}_n =: \tilde{\mathcal{U}}$

THE $S_{\mathbb{N}}$-REPR

UNDER CANONICAL INCLUSIONS

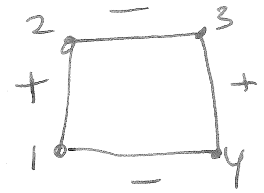
RECOVER $\mathcal{U}_T = \{ u \in \tilde{\mathcal{U}} \mid S_{\mathbb{N}-T}$ FIXES $u \}$

MORE TECHNICAL CONDITION: (SATURATION NOT QUITE ENOUGH)

IF $0 \to X \to M \to U \to 0$ (M FREE, U SATURATED) THEN X IS "2-SATURATED":

DEF: X is 2-SATURATED if it's SATURATED AND FOR ANY $S, S' \subseteq T$, $X_T \cap (M_S + M_{S'}) = X_S + X_{S'}$ AS SUBMODULES OF M_T (OR X_T)

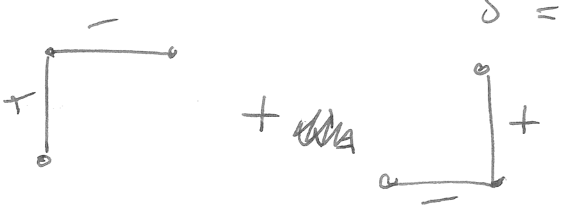
EX: $U \leq M(\mathrm{triv}_2)$ (FREE ON EDGES OF GRAPH)



$T = \{1,2,3,4\}$

$S = \{1,2,3\}$

$S' = \{1,3,4\}$

$u =$  $u \in M_S + M_{S'}$

BUT $u \notin U_S + U_{S'}$.

PROP: IF $0 \to X \to M \to U \to 0$ AND U is SATURATED, THEN X is 2-SATURATED.

(ALTERNATIVE CHARACTERIZATION: $\forall \ell$, $H_1(W_{+\ell} V)_3 = 0$)

DEF: Y is ℓ-SATURATED if $(\ell-1)$-SATURATED AND $Y_T \cap (M_{S_1} + \dots + M_{S_\ell}) = Y_{S_1} + \dots + Y_{S_\ell}$.

DEF: WE SAY U is ℓ-SATURATED ABOVE N ($\geq N$) IF THE ABOVE HOLDS WHEN $|S_i| \geq N$.

PROP B: IF $X \subseteq M(d)$ is $(d+1)$-SATURATED $\geq N$ THEN X is GENERATED BY X_N. (OR X_ℓ FOR $\ell \leq N$)

PF: $T=[n]$, $S_1=[n]-\{1\}$, ... $S_{d+1}=[n]-\{d+1\}$

EVERY INJECTION $f:[d]\hookrightarrow[n]$ FACTORS THROUGH SOME S_i $\Longrightarrow$ $M_T = M_{S_1}+\dots+M_{S_{d+1}}$.

$X_T = X_T\cap(M_{S_1}+\dots+M_{S_{d+1}}) = X_{S_1}+\dots+X_{S_{d+1}}$

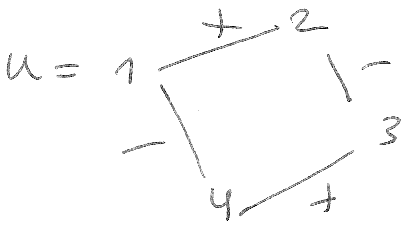
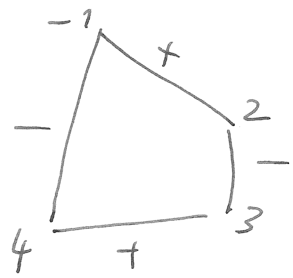
i.e. X_n IS SPANNED BY MAPS FROM $X_{n-1}\to X_n$.

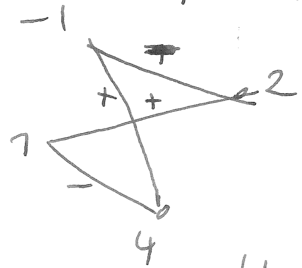
PROP C: IF $0\to V\to M\to W\to 0$, W GENERATED IN deg K AND $0\to U\to M'\to V\to 0$, V GENERATED IN deg D, THEN U is k-SATURATED$_{\geq N}$ FOR ALL k WITH $N=D+K+1$.

~~SKETCH FOR WORD~~

INTRINSIC DEFINITION OF k-SATURATED: $\ker J_1\cdots J_k = \ker J_1+\dots+\ker J_k$

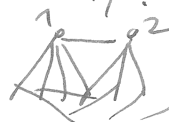
$J_1\in \mathbb{Z}S_{N\cup\{-1\}}$, $J_1 = \mathrm{Id}-(1\ {-1})$
↑ TRANSPOSITION

FOR $u=$  , $J_1u = u-$ 

$=$ 

SATURATION $\Longrightarrow$ $\ker J_1 = \mathrm{im}(U_{\{2,\dots,n\}}\to U_{\{1,\dots,n\}})$
AS SUBSETS OF $U_{\{1,\dots,n\}}$

$J_2 = (\mathrm{id})-(2\ {-2})$. J_1, J_2 COMMUTE.

J_1J_2 ANIHILATES ANYTHING NOT INVOLVING 2, ALSO NOT INVOLVING 1. THE EDGE 12 

$J_1J_2 = J_2J_1$ $\Longrightarrow \ker J_1J_2 = M_{S_1}+M_{S_2}$