LECTURE II (TOM CHURCH)

RECALL: FI = FINITE SETS AND INJECTIONS

$V: FI \to \mathbb{Z}\text{-Mod}$ FI-MODULE

$M(d)$ IS "FREE" FI-MODULE $S \mapsto \operatorname{Hom}_{FI}([d], S) = \mathbb{Z}\{(s_1, \dots, s_d) \text{ DISTINCT IN } S\}$

$M(A) = A \otimes_{\mathbb{Z}S_d} M(d)$ FOR $\mathbb{Z}S_d$-MODULE A

EXERCISE FROM LAST TIME:

TRIVIAL S_2-ACTION

$\langle e_1 + e_2 \rangle = \mathbb{Z}$ $\{(a,b,c) \mid a+b+c \text{ even}\} = \mathbb{Z}^3_{ev}$

$$\begin{tikzcd}
0 \arrow[d] & & & & & & \\
V \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}^3_{ev} \arrow[r] \arrow[d] & \mathbb{Z}^4_{ev} \arrow[r] \arrow[d] & \cdots \\
\mathbb{Z}[S] = M(1) \arrow[d] & 0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}^2 \arrow[r] \arrow[d] & \mathbb{Z}^3 \arrow[r] \arrow[d] & \mathbb{Z}^4 \arrow[r] \arrow[d] & \cdots \\
W \arrow[d] & 0 \arrow[r] & \mathbb{Z} \arrow[r, shift left, "id"] \arrow[r, shift right, "-id"'] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & {} \\
0 & & & & & &
\end{tikzcd}$$

WHAT ARE GENERATORS FOR V?

1) $e_1 + e_2 \in V_2 \rightsquigarrow e_i + e_j \in V_n$

2) $2e_1 \in V_3 \longrightarrow 2e_i \in V_n$

UNNECESSARY:

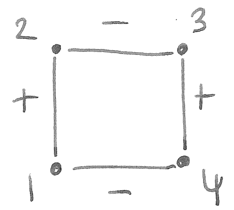
$$\begin{tikzcd}[row sep=small]
 & e_1 + e_2 \in V_3 \\
e_1 + e_2 \arrow[ur] \arrow[r] \arrow[dr] & -(e_2 + e_3) \in V_3 \\
 & e_1 + e_3 \in V_3
\end{tikzcd}$$

$$\overline{\quad 2e_1 \in V_3 \quad}$$

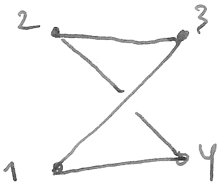
$$\begin{tikzcd}[column sep=small]
0 \arrow[d] & & & & & & \\
\mathcal{U} \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z}^2 \arrow[r] \arrow[d] & \mathbb{Z}^5 \arrow[r] \arrow[d] & \cdots \\
\mathbb{Z}\left[\binom{5}{2}\right] = M(\mathrm{triv}_2) \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] \arrow[loop above, "\text{TRIVIAL}"] & \mathbb{Z}\left[\binom{3}{2}\right] \simeq \mathbb{Z}^3 \arrow[r] \arrow[d] & \mathbb{Z}\left[\binom{4}{2}\right] \simeq \mathbb{Z}^6 \arrow[r] \arrow[d] & \mathbb{Z}\left[\binom{5}{2}\right] \simeq \mathbb{Z}^{10} \arrow[r] \arrow[d] & \cdots \\
V \arrow[d] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}^3_{ev} \arrow[r] & \mathbb{Z}^4_{ev} \arrow[r] & \mathbb{Z}^5_{ev} \arrow[r] & \cdots \\
0 & & & & & &
\end{tikzcd}$$

THE MAP $M(\mathrm{triv}_2) \longrightarrow V$

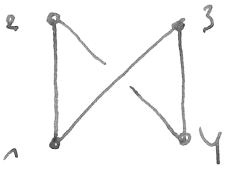
$\{e_i, e_j\} \longmapsto e_i + e_j$



$e_{12} - e_{23} + e_{34} - e_{41} \in \ker\big(\underset{\mathcal{U}_4}{\overset{\shortparallel}{\mathbb{Z}^2}} \longrightarrow \mathbb{Z}\left[\binom{4}{2}\right]\big)$



$e_{14} - e_{42} + e_{23} - e_{31}$



$e_{13} - e_{34} + e_{42} - e_{21}$

0

GENERATE THE KERNEL AND SUM TO 0

THE ACTION OF S_4 IS WHAT IT IS ... $\mathbb{Z}^2_{\boxplus}$

RELIEF: $\mathcal{U}$ IS GENERATED BY $\mathcal{U}_4$.

TODAY: THEOREM WITH ELLENBERG WHICH ASSUMES THAT $\mathcal{U}$ IS GENERATED BY $\mathcal{U}_4$.

MINIMAL GENERATORS

DEF: $H_0^{FI}(V)$ IS A GRADED $\bigoplus S_n$-MODULE, i.e.

$H_0^{FI}(V) = \bigoplus H_0^{FI}(V)_n$, $H_0^{FI}(V)_n$ is A $\mathbb{Z}S_n$-MODULE.

$H_0^{FI}(V)_n = V_n / \langle$ IMAGES OF ALL MAPS $f_*: V_{n-1} \to V_n \rangle$

(EQUIV. ALL MAPS $f_*: V_k \to V_n$, $k<n$)

(THIS IS THE LATCHING SPACE FOR V ~~FOR~~ AT [n].)

EX: $H_0^{FI}(M(1))$: $0 \quad \mathbb{Z} \quad 0 \quad 0 \quad \ldots$ $\qquad 0 \to \underset{\langle e_1 \rangle}{\mathbb{Z}} \to \mathbb{Z}^2 \to \mathbb{Z}^3 \to \cdots$

EX: $H_0^{FI}(\mathbb{Z}[\binom{S}{2}])$: $0 \quad 0 \quad \mathbb{Z}_{triv} \quad 0 \quad 0 \quad \ldots$

MORE GENERALLY, $H_0^{FI}(M(d)) = 0 \quad 0 \ldots 0 \quad \mathbb{Z}S_d \quad 0 \ldots 0 \ldots$

$H_0^{FI}(M(A)) = 0 \ldots 0 \quad A \quad 0 \ldots 0 \ldots$

EX: $H_0^{FI}(W)$: $0 \quad \mathbb{Z} \quad 0 \quad 0 \quad \ldots$

$H_0^{FI}(V)$ $\quad 0 \quad 0 \quad \mathbb{Z}_{triv} \quad 0 \quad \ldots$

$H_0^{FI}(U)$ $\quad 0 \quad 0 \quad 0 \quad 0 \quad \mathbb{Z}^2_{\boxplus} \quad 0 \quad \ldots$

$\to H_0$ RECORDS A MINIMAL GENERATING SET.

COMPARE WITH GRADED $k[T]$-MODULES (WITH $|T|=1$):

GRADED $k[T]$-MODULE $M \rightsquigarrow H_0^{k[T]}(M) = M/TM$

$H_0^{k[T]}(M)_n = M_n / \mathrm{im}(T: M_{n-1} \to M_n)$

$(M_0 \xrightarrow{T} M_1 \xrightarrow{T} M_2 \to \cdots)$

DEF: $H_i^{FI}(V)$ IS DEFINED AS THE LEFT-DERIVED FUNCTORS OF H_0^{FI}

REM: $H_0^{FI}(V) = K \otimes_{FI} V$ WHERE $K = \bigoplus_{n=0}^{\infty} \mathbb{Z} S_n$ ~~WHERE~~ AND

ALL MAPS $K_n \to K_m$ $n \neq m$ ARE 0

$H_i^{FI}(V) = \mathrm{Tor}_i^{FI}(K, V)$ (K is AN $FI \times FI^{op}$-MODULE $\to H_i^{F}(V)$ is AGAIN AN FI-MODULE)

IN PARTICULAR, $H_i^{FI}(M(d)) = 0$ $i > 0$ AS $M(d)$ IS PROJECTIVE. ALSO $H_i^{FI}(M(A)) = 0$ $i > 0$: ONE WOULD RESOLVE A AS S_d-MODULES, THEN M OF THAT RESOLUTION GIVES A RESOLUTION OF $M(A)$.

IN THE EXAMPLE: $0 \to V \to M(1) \to W \to 0$

$\leadsto$ LONG EXACT SEQUENCE

$$\underbrace{\cancel{H_1(M(1))}}_{0} \to \boxed{H_1(W) \xrightarrow[\text{iso}]{} H_0(V)} \to H_0(M(1)) \xrightarrow{\cong} H_0(W)$$

~~MI~~ MINIMAL GENERATORS FOR ARE MINIMAL RELATIONS FOR W

$\to$ THAT IS WHAT H_1^{FI} COMPUTES

MORE GENERALLY, H_i^{FI} COMPUTES MINIMAL ~~RESOLUTIONS~~ RELATIONS BETWEEN RELATIONS —

EXAMPLE: COMPUTE $H_i^{FI}(\mathbb{Z}_0)$ — $\mathbb{Z} \to 0 \to 0 \to \cdots$

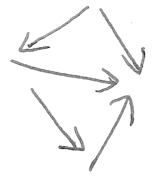
BY BUILDING A RESOLUTION

SIGN REP

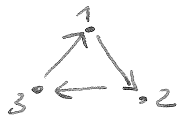
$$
\begin{tikzcd}
 & & & & 0 \arrow[r] \arrow[d] & \mathbb{Z}_\varepsilon \arrow[r] \arrow[d, "?"'] & \mathbb{Z}^4 \arrow[r] \arrow[d, "?"'] & {} \\
M(\varepsilon_2) \arrow[d] & 0 \arrow[r] & 0 \arrow[r] & \underset{\substack{\| \\ \mathbb{Z}_\varepsilon}}{\langle e_1 - e_2 \rangle} \arrow[r] \arrow[d] & \mathbb{Z}^3 \arrow[r] \arrow[d] & \mathbb{Z}^6 \arrow[r] \arrow[d] & \cdots \\
M(1) \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "\cong"] & \mathbb{Z}^2 \arrow[r] \arrow[d, "{(a,b) \mapsto a+b}"] & \mathbb{Z}^3 \arrow[r] \arrow[d] & \mathbb{Z}^4 \arrow[r] & \cdots \\
M(0) \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "\cong"] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}\ (\text{TRIV}) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \cdots \\
\mathbb{Z}_0 & \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & \cdots
\end{tikzcd}
$$

$M(\varepsilon_2)$ = DIRECTED EDGES

(A SIGN COMES WHEN TURNING AN EDGE)



$i \to j \;\mapsto\; e_i - e_j$

$$\ker\Big(\underset{\substack{\| \\ M(\varepsilon_2)_3}}{\mathbb{Z}^3} \longrightarrow \underset{\substack{\| \\ M(1)_3}}{\mathbb{Z}^3}\Big) = \Big\langle$$  $$\Big\rangle$$

IN FACT, THE ABOVE COMES FROM THE REDUCED CHAIN CPX $C(\Delta^{\bullet-1})$:

$$
\begin{array}{ccc}
\downarrow & & \downarrow \\
M(\varepsilon_2) & \sim & C_2(\Delta^{\bullet-1}) \\
\downarrow & & \downarrow \\
M(1) & \sim & C_1(\Delta^{\bullet-1}) \\
\downarrow & & \downarrow \\
M(0) & \sim & C_0(\Delta^{\bullet-1}) \\
\downarrow & & \downarrow \\
\mathbb{Z}_0 & \sim & C_{-1}(\Delta^{\bullet-1})
\end{array}
$$

EXACT BECAUSE $\Delta^{\bullet-1}$ IS CONTRACTIBLE

⚠ Δ^{n-1} IS CONTRACTIBLE $\forall n \geq 1$

BUT WHEN $n=0$, $\Delta^{n-1} = \emptyset$.

APPLYING H_0, GET

$$\begin{matrix} 0 & 0 & 0 & \varepsilon_3 & 0 \\ 0 & 0 & \varepsilon_2 & 0 & 0 \\ 0 & \varepsilon_1 & 0 & 0 & 0 \\ \varepsilon_0 & 0 & 0 & 0 & 0 \end{matrix}$$

$\rightsquigarrow H_i(Z_0) = \underbrace{\varepsilon_i}$ SIGN REPRESENTATION OF S_i.

NOTE: A $\mathbb{Q}S_d$-REP WITH $\dim A = a$, THEN $\dim(M(A)) = \binom{n}{d} a$

$\dim M(d) = \binom{n}{d} d!$

DEF: $\deg H_i^{FI}(V) = \sup\{n \mid H_i^{FI}(V)_n \neq 0\}$

THM (CHURCH, ELLENBERG) $\deg H_i^{FI}(W) \leq \deg H_0^{FI}(W) + \deg H_1^{FI}(W) - 1 + i$

$\forall i \geq 2$

EXAMPLE: HAD ~~deg~~ $\deg H_0(W) = 1$

$\deg H_0(V) = \deg H_1(W) = 2$

$\deg H_0(U) = \deg H_1(V) = \deg H_2(W) = 4$

$\Rightarrow \deg H_i^{FI}(W) \leq 1 + 2 - 1 + i = i + 2 \quad \to \quad \deg H_2(W) \leq 4$

$\deg H_3(W) \leq 5$

NOTE: $H_i(V) = H_{i+1}(W)$ BUT THE THEOREM GIVES A WORSE BOUND FOR $H_i(V)$ THAN FOR $H_{i+1}(W)$