HOMOLOGICAL STABILITY VIA FI-MODULES

TOM CHURCH

LECTURE I

MOTIVATING GOAL: CONSIDER $GL_n \mathbb{Z}$ AND $\Gamma_n(p) = \ker(GL_n\mathbb{Z} \to GL_n \mathbb{Z}/p\mathbb{Z})$

THM (CHARNEY, MAAZEN)

FOR $n > 3i$, $H_i(GL_n\mathbb{Z}) \xrightarrow{\cong} H_i(GL_{n+1}\mathbb{Z})$

THM (WE WISH)

$$H_i(\Gamma_n(p)) \xrightarrow{\cancel{\cong}} H_i(\Gamma_{n+1}(p))$$

TRUE WITH RATIONAL COEFFICIENTS BUT NOT INTEGRALLY

$\Gamma_n(p) \twoheadrightarrow \mathfrak{sl}_n \mathbb{F}_p$ (TRACELESS $n\times n$-MATRICES) ABELIAN GROUP

$I + pA \longmapsto A \bmod p$

$\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow H_1(\Gamma_n(p)) \geq \mathfrak{sl}_n \mathbb{F}_p$ (IN FACT EQUAL)

$\Rightarrow$ CANNOT STABILIZE.

INCLUSION $GL_n(\mathbb{Z}) \to GL_{n+1}(\mathbb{Z})$

$$A \longmapsto \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right)$$

QUESTION: WHY NOT $A \longmapsto \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & A \end{array}\right)$?

$\to$ THEY ARE CONJUGATE IN $GL_{n+1}(\mathbb{Z}) \Rightarrow$ SAME MAP IN H_*:

$$\begin{tikzcd} H_i(GL_n(\mathbb{Z})) \arrow[r] \arrow[dr] & H_i(GL_{n+1}\mathbb{Z}) \\ & H_i(GL_{n+1}\mathbb{Z}) \arrow[u, "id"] \end{tikzcd}$$

VERY IMPORTANT FACT IN THE PROOF OF H_*-STABILITY!

BUT THE CONJUGATION DOESN'T HAPPEN IN $\Gamma_{n+1}(p)$!

$\Gamma_2(p) \hookrightarrow \Gamma_3(p)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \mathrm{I}\begin{pmatrix} a & b & \\ c & d & \\ & & 1 \end{pmatrix} \qquad \mathrm{II}\begin{pmatrix} 1 & & \\ & a & b \\ & c & d \end{pmatrix} \qquad \mathrm{III}\begin{pmatrix} a & & b \\ & 1 & \\ c & & d \end{pmatrix}$$

$$\Big\downarrow C\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathrm{IV}\begin{pmatrix} d & c & \\ b & a & \\ & & 1 \end{pmatrix} \qquad \mathrm{V}\begin{pmatrix} a & b & -b \\ 2c & 2d-1 & 2-2d \\ c & d-1 & 2-d \end{pmatrix} \qquad \mathrm{VI}\begin{pmatrix} a & 4ab & -9b \\ -2c & 1-8(d-1) & 18(d-1) \\ -9c & 360(d-1) & 1+81(d-1) \end{pmatrix}$$

(CONJUGATE TO 1 WHEN $p=3$)

....

WE WILL ONLY NEED TO LOOK AT THE FIRST 3 TYPES OF INCLUSIONS.

<u>DEF</u>: FOR ANY SUBSET $J \subset [n] = \{1, .., n\}$, DEFINE

$$\Gamma_n^J(p) = \left\{ \text{MATRICES IN } GL_n\mathbb{Z} \text{ THAT ARE CONGRUENT TO id mod } p \text{ AND } \underline{\text{EQUAL}} \text{ TO id IN } j^{th} \text{ ROW AND COLUMN } \forall j \in J \right\}$$

<u>EQ</u>: $\Gamma_3^{\{2\}}(p) = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix} \in \Gamma_3(p) \right\}$

<u>NOTE</u>: $\Gamma_n^J(p)$ ISOMORPHIC TO $\Gamma_{n-|J|}(p)$, BUT THIS MAKES THE INCLUSION $\Gamma_n^J(p) \hookrightarrow \Gamma_p(p)$ UNAMBIGUOUS.

<u>THM</u> (CHURCH - ELLENBERG, D'APRÈS PUTMAN)

CONSIDER THE n DIFFERENT INCLUSIONS

$$\varphi_j : H_i(\Gamma_n^{\{j\}}(p); \mathbb{Z}) \longrightarrow H_i(\Gamma_n(p); \mathbb{Z}) \quad \text{FOR } j = 1, .., n$$

FOR $n \geq 9.2^i - 7$ WE HAVE

a) $H_i(\Gamma_n(p);\mathbb{Z})$ IS SPANNED BY THE IMAGES OF $\varphi_1,\dots,\varphi_n$.

b) WE HAVE A PRESENTATION

$$H_i(\Gamma_n(p);\mathbb{Z}) = \frac{\bigoplus_{j=1}^{n} H_i(\Gamma_n^{\{j\}}(p);\mathbb{Z})}{\mathrm{im}\left(\bigoplus_{j,k=1}^{n} H_i(\Gamma_n^{\{j,k\}}(p);\mathbb{Z})\right)}$$

WHERE $H_i(\Gamma_n^{\{j,k\}}(p)) \xrightarrow[(id,-id)]{} H_i(\Gamma_n^{\{j\}}(p)) \oplus H_i(\Gamma_n^{\{k\}}(p))$

(PROVED IN PUTMAN '12 WITH COEFFICIENTS FIELD OF char 0 OR OF char $p \geq 9.2^{i-1}-3$)

PLAN OF THE LECTURES: PROVE THIS THEOREM.

REFERENCES: [CEF] CHURCH, ELLENBERG, FARB arXiv 1204.4533
[CEFN] CHURCH, ELLENBERG, FARB, NAGPAL, — 12.10.1854
[CE] CHURCH, ELLENBERG (IN PREPARATION)

FI-MODULES

DEF: FI = CATEGORY OF FINITE SETS AND INJECTIONS

AN FI-MODULE V IS A FUNCTOR $V: FI \to \mathbb{Z}\text{-Mod}$

FI HAS SKELETON $[n]=\{1,\dots,n\}$ FOR $n\in\mathbb{N}$, SO V DETERMINES ABELIAN GROUPS $V_n = V([n])$
AND $\forall f:[n]\hookrightarrow[m]$, HAVE $V_n \longrightarrow V_m$

$\mathrm{End}_{FI}([n]) = S_n \longrightarrow V_n$ IS AN S_n-REPRESENTATION $= \mathbb{Z}S_n$-MODULE.

EX: $\mathbb{Z}[S]$: $\quad V: S \longmapsto \mathbb{Z}[S] \qquad V_n = \mathbb{Z}^n = \langle e_1, \dots, e_n \rangle$

EX: $\mathbb{Z}\left[\binom{S}{3}\right] \qquad S \longmapsto \mathbb{Z}\left[\binom{S}{3}\right] \qquad V_n = \mathbb{Z}^{\binom{n}{3}} = \langle e_{123}, e_{124}, \dots \rangle$

3-ELEMENT SUBSETS

EX: $\mathbb{Z}_0$ $\qquad \begin{cases} \emptyset \longmapsto \mathbb{Z} \\ S \longmapsto 0 & S \neq \emptyset \end{cases}$ $\qquad \begin{matrix} V_0 & & V_1 & & V_2 & & \cdots \\ \mathbb{Z} & \to & 0 & \to & 0 & \to & \cdots \end{matrix}$

EX: $H^i(\mathrm{Conf}(M)) \qquad S \longmapsto H^i(\mathrm{Emb}(S, M)) \qquad V_n = H^i(\mathrm{Conf}_n(M))$

LABELED CONFIGURATIONS IN M

EX: $H_i(\Gamma(p)) \qquad \Gamma(p): FI \longrightarrow \text{GROUPS}$

$S \longmapsto \mathrm{Aut}(\mathbb{Z}[S] \text{ CONGRUENT TO id mod } p) \simeq \Gamma_{|S|}(p)$

$$(H_i(\Gamma(p)))_n \simeq H_i(\Gamma_n(p); \mathbb{Z})$$

(REPLACING $\Gamma(p)$ BY GL. GIVES A "NOT SO INTERESTING" FI-MODULE/GROUP)

DEF: A MAP OF FI-MODULES $\varphi: V \to W$ IS A NATURAL TRANSFORMATION OF FUNCTORS i.e.,

$$\begin{array}{ccc} V_n & \xrightarrow{\varphi_n} & W_n \\ \downarrow f_* & & \downarrow f_* \\ V_m & \xrightarrow[\varphi_m]{} & W_m \end{array} \qquad \varphi_m(f_* v) = f_*(\varphi_n(v)) \quad \forall f, v$$

EX: $\varphi: \mathbb{Z}[S] \longrightarrow W$ $\qquad$ EVERY $e_i \in V_n = \mathbb{Z}^n$ IS IMAGE OF $f_*(e_1)$ OF $e_1 \in V_1 = \mathbb{Z}$

$\varphi_n(e_i) = \varphi_n(f_*(e_1)) = f_*(\varphi_1(e_1)) \quad , \varphi_1(e_1) \in W_1$

$\rightsquigarrow \mathrm{Hom}(\mathbb{Z}[S], W) = W_1$

Def: $M(d)$ is the FI-module

$$S \longmapsto \mathbb{Z}\,\mathrm{Hom}_{FI}([d], S)$$

(Yoneda lemma) $\mathrm{Hom}_{FI\text{-Mod}}(M(d), W) \simeq W_d$

NOTE:
$$\begin{array}{ccc} FI & & FI\text{-Mod} \\ \cup & & \downarrow \\ */S_d & & \mathbb{Z}S_d\text{-Mod} \\ \cup & & \downarrow \\ * & & \mathbb{Z}\text{-Mod} \end{array}$$

$\mathbb{Z}[S] \simeq M(1)$

$\mathbb{Z} \simeq M(0)$

$$\begin{tikzcd} FI\text{-Mod} \arrow[d, "\pi_d"'] \\ \mathbb{Z}\text{-Mod} \arrow[u, bend left, "\pi_*"] \end{tikzcd} \qquad \begin{array}{c} V \\ \downarrow \\ V([d]) \end{array} \qquad\qquad \begin{tikzcd} FI\text{-Mod} \arrow[d] \\ \mathbb{Z}S_d\text{-Mod} \arrow[u, bend left, "\pi_*"] \end{tikzcd} \qquad \begin{array}{c} V \\ \downarrow \pi \\ V([d]) \end{array}$$

LEFT KAN EXT

$\pi_*(\mathbb{Z}) = M(d)$.

$\pi_*(\mathbb{Z}S_d) = M(d)$

$M(d): S \mapsto \mathbb{Z}[(s_1, \dots, s_d) \mid \text{distinct elements of } S]$

$\mathrm{Inj}([d], S)$

$\pi_*(\mathrm{triv}_{S_3}) = \mathbb{Z}\left[\binom{S}{3}\right]$

$\left(\mathrm{Hom}_{FI}(\mathbb{Z}\binom{S}{3}, W) = (W_3)^{S_3}\right)$

Def: An FI-module V is finitely generated if $\exists$ surjection $\bigoplus_{i=1}^{\ell} M(d_i) \twoheadrightarrow V$

with "generators" in V_{d_i}.

EX:

$$\begin{array}{ccccccccccc} & W_0 & & W_1 & & W_2 & & W_3 & & \dots \\ W = & 0 & \to & \mathbb{Z} & \overset{id}{\underset{-id}{\rightrightarrows}} & \mathbb{Z} & \to & \mathbb{Z}/2\mathbb{Z} & \to & \mathbb{Z}/2\mathbb{Z} & \to \mathbb{Z}/2\mathbb{Z} \to \cdots \end{array}$$

$\{1\} \hookrightarrow \{1,2\}$

Generated by W_1 $\rightsquigarrow$ $\mathbb{Z}[S] = M(1) \twoheadrightarrow W$

$$0 \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z}^3 \to \mathbb{Z}^4 \to \mathbb{Z}^5 \to \cdots$$

Exercice: What's the kernel FI-module:

$$0 \to V \to M(1) \twoheadrightarrow W \to 0$$

Find (minimal?) generators for V.