

Homological Stability via FI-Modules

(2)

Jon Church

LECTURE I

MOTIVATING GOAL: Consider $GL_n \mathbb{Z}$ and $\Gamma_n(p)$

$$\ker(GL_n \mathbb{Z} \rightarrow GL_n \mathbb{Z}/p\mathbb{Z})$$

THM (CHERVEN, MAZEN)

$$\text{For } n > 3i, H_i(GL_n \mathbb{Z}) \xrightarrow{\cong} H_i(GL_{n+1} \mathbb{Z})$$

THM (WE WISH)

$$H_i(\Gamma_n(p)) \not\xrightarrow{\cong} H_i(\Gamma_{n+1}(p))$$

TRUE WITH RATIONAL COEFFICIENTS BUT NOT INTEGRALLY

$$\Gamma_n(p) \twoheadrightarrow \text{sl}_n \mathbb{F}_p \quad (\text{TRACES } n \times n\text{-MATRICES})$$

ABELIAN GROUP

$$I + pA \longmapsto A \pmod{p}$$

$$\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow H_1(\Gamma_n(p)) \cong \text{sl}_n \mathbb{F}_p$$

(IN FACT EQUAL)

\Rightarrow CANNOT STABILIZE.

INCLUSION $GL_n(\mathbb{Z}) \rightarrow GL_{n+1}(\mathbb{Z})$

$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

QUESTION: WHY NOT $A \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$?

\rightarrow THEY ARE CONJUGATE IN $GL_{n+1}(\mathbb{Z}) \Rightarrow$ SAME MAP IN H_* :

$$\begin{array}{ccc} H_i(GL_n(\mathbb{Z})) & \longrightarrow & H_i(GL_{n+1}(\mathbb{Z})) \\ & & \uparrow \text{id} \\ & \searrow & H_i(GL_{n+1}(\mathbb{Z})) \end{array}$$

VERY IMPORTANT FACT
IN THE PROOF OF
 H_* -STABILITY!

BUT THE CONJUGATION DOESN'T HAPPEN IN $\Gamma_{n+1}(p)$!

$$\Gamma_2(p) \hookrightarrow \Gamma_3(p)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \text{I} \begin{pmatrix} a & b & \\ c & d & 1 \end{pmatrix} \quad \text{II} \begin{pmatrix} 1 & & \\ & a & b \\ & c & d \end{pmatrix} \quad \text{III} \begin{pmatrix} a & & b \\ & 1 & \\ c & & d \end{pmatrix}$$

$$\downarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{IV} \begin{pmatrix} d & c & \\ b & a & \\ & & 1 \end{pmatrix} \quad \text{V} \begin{pmatrix} a & b & -b \\ 2c & 2d-1 & 2-2d \\ c & d-1 & 2-d \end{pmatrix} \quad \text{VI} \begin{pmatrix} a & 4ab & -9b \\ -2c & 1-8(d-1) & 18(d-1) \\ -9c & 36c(d-1) + 81(d-1) & \end{pmatrix}$$

...

(CONJUGATE TO 1
WHEN $p=3$)

WE WILL ONLY NEED TO LOOK AT THE FIRST 3 TYPES OF INCLUSIONS.

Def: For any subset $J \subset [n] = \{1, \dots, n\}$, define

$$\Gamma_n^J(p) = \left\{ \text{MATRICES IN } GL_n \mathbb{Z} \text{ THAT ARE CONGRUENT TO id mod } p \text{ AND EQUAL TO id IN } j^{\text{th}} \text{ ROW AND COLUMN } \forall j \in J \right\}$$

Ex: $\Gamma_3^{\{2,3\}}(p) = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix} \in \Gamma_3(p) \right\}$

NOTE: $\Gamma_n^J(p)$ IS ISOMORPHIC TO $\Gamma_{n-|J|}(p)$, BUT THIS MAKES THE INCLUSION $\Gamma_n^J(p) \hookrightarrow \Gamma_p(p)$ UNAMBIGUOUS.

THM (CHURCH - EISENBURG, D'APRÈS FURMAN)

CONSIDER THE n DIFFERENT INCLUSIONS

$$\varphi_j: H_i(\Gamma_n^{\{j\}}(p); \mathbb{Z}) \longrightarrow H_i(\Gamma_n(p); \mathbb{Z}) \quad \text{FOR } j=1, \dots, n$$

FOR $n \geq 9 \cdot 2^i - 7$ WE HAVE

a) $H_i(\Gamma_n(p); \mathbb{Z})$ is spanned by the images of $\varphi_1, \dots, \varphi_n$.

b) we have a presentation

$$H_i(\Gamma_n(p); \mathbb{Z}) = \frac{\bigoplus_{j=1}^n H_i(\Gamma_n^{\{j\}}(p); \mathbb{Z})}{\text{im}\left(\bigoplus_{j, k=1}^n H_i(\Gamma_n^{\{j, k\}}(p); \mathbb{Z})\right)}$$

where $H_i(\Gamma_n^{\{j, k\}}(p)) \xrightarrow{(\text{id}, -\text{id})} H_i(\Gamma_n^{\{j\}}(p)) \oplus H_i(\Gamma_n^{\{k\}}(p))$

(Proved in Putman '12 with coefficients field of char 0 or of char $p \geq 9 \cdot 2^{i-1} - 3$)

Plan of the lectures: Prove this theorem.

- [CEF] REFERENCES: Church, Eilenberg, Farb arXiv 1204.4533
- [CEFN] Church, Eilenberg, Farb, Nagpal, — 12.10.1854
- [CE] Church, Eilenberg (in preparation)

FI-MODULE

DEF: FI = CATEGORY OF FINITE SETS AND INJECTIONS

AN FI-MODULE V IS A FUNCTOR $V: \text{FI} \rightarrow \mathbb{Z}\text{-Mod}$

FI HAS SKELETON $[n] = \{1, \dots, n\}$ FOR $n \in \mathbb{N}$, SO V DETERMINES ABELIAN GROUPS $V_n = V([n])$

AND $\forall f: [n] \hookrightarrow [m]$, HAVE $V_n \rightarrow V_m$

$\text{End}_{\text{FI}}([n]) = S_n \rightarrow V_n$ IS AN S_n -REPRESENTATION = $\mathbb{Z}S_n$ -MODULE.

EX: $\mathbb{Z}[S]$: $V: S \mapsto \mathbb{Z}[S]$ $V_n = \mathbb{Z}^n = \langle e_1, \dots, e_n \rangle$
EX: $\mathbb{Z}\left[\binom{S}{3}\right]$ $S \mapsto \mathbb{Z}\left[\binom{S}{3}\right]$ $V_n = \mathbb{Z}^{\binom{n}{3}} = \langle e_{123}, e_{124}, \dots \rangle$
 \uparrow 3-element subsets

EX: \mathbb{Z}_0 $\left\{ \begin{array}{l} \emptyset \mapsto \mathbb{Z} \\ S \mapsto 0 \quad S \neq \emptyset \end{array} \right.$ $V_0 \quad V_1 \quad V_2 \quad \dots$
 $\mathbb{Z} \rightarrow 0 \rightarrow 0 \rightarrow \dots$

EX: $H^i(\text{Conf}(M))$ $S \mapsto H^i(\text{Emb}(S, M))$ $V_n = H^i(\text{Conf}_n(M))$
 LABELED CONFIGURATIONS IN M

EX: $H_i(\Gamma(p))$ $\Gamma(p): \text{FI} \rightarrow \text{Group}$
 $S \mapsto \text{Aut}(\mathbb{Z}[S] \text{ congruent to id mod } p) \cong \Gamma_{|S|}(p)$

$(H_i(\Gamma(p)))_n \cong H_i(\Gamma_n(p); \mathbb{Z})$

(REPLACING $\Gamma(p)$ BY GL . GIVES A "NOT SO INTERESTING" FI-MODULE)
 GROUP

DEF: A MAP OF FI-MODULES $\varphi: V \rightarrow W$ IS A NATURAL TRANSFORMATION OF FUNCTORS I.E.,

$$\begin{array}{ccc} V_n & \xrightarrow{\varphi_n} & W_n \\ \downarrow f_* & & \downarrow f_* \\ V_m & \xrightarrow{\varphi_m} & W_m \end{array} \quad \varphi_m(\varphi_* v) = f_* (\varphi_n(v)) \quad \forall f_{1,2}$$

EX: $\varphi: \mathbb{Z}[S] \rightarrow W$ EVERY $e_i \in V_n = \mathbb{Z}^n$ IS MADE OF $f_*(e_i)$ OF $e_i \in V_1 = \mathbb{Z}$
 $\varphi_n(e_i) = \varphi_n(f_*(e_i)) = f_*(\varphi_1(e_i))$, $\varphi_1(e_i) \in W_1$

$\rightarrow \text{Hom}(\mathbb{Z}[S], W) = W_1$

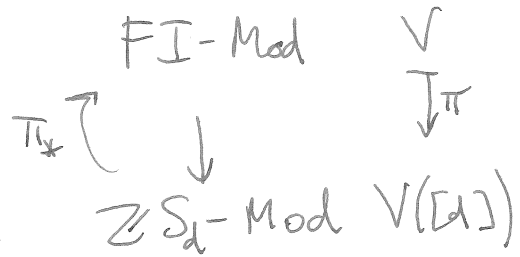
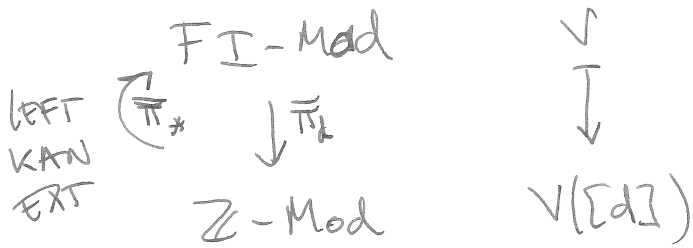
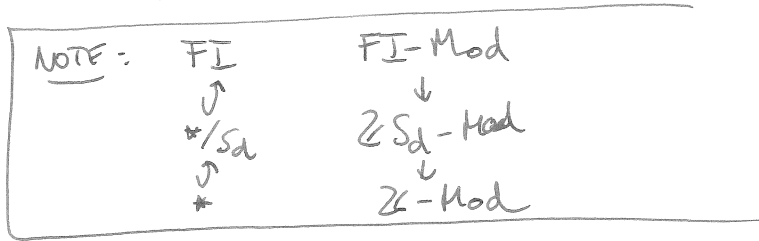
Def: $M(d)$ is the FI-module

$$S \mapsto \mathbb{Z} \text{Hom}_{\text{FI}}([d], S)$$

(YONEDA LEMMA) $\text{Hom}_{\text{FI-Mod}}(M(d), W) \cong W_d$

$$\mathbb{Z}[S] \cong M(1)$$

$$\mathbb{Z} \cong M(0)$$



$$\bar{\pi}_\#(\mathbb{Z}) = M(d)$$

$$\pi_\ast(\mathbb{Z}S_d) = M(d)$$

$$M(d): S \mapsto \mathbb{Z}[(s_1, \dots, s_d) \mid \text{DISTINCT EVENTS OF } S]$$

$$\text{Fin}([d], S)$$

$$\pi_\ast(\text{triv}_{S_3}) = \mathbb{Z} \left[\binom{S}{3} \right]$$

$$(\text{Hom}_{\mathbb{Z}}(\mathbb{Z} \binom{S}{3}, W) = (W_3)^{S_3})$$

Def: An FI-module V is finitely generated if

$$\exists \text{ SURJECTION } \bigoplus_{i=1}^e M(d_i) \rightarrow V$$

with "generators" in V_{d_i} .

EX:

$$W = 0 \rightarrow \mathbb{Z} \xrightarrow[\text{-id}]{\text{id}} \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \dots$$

$\{1\} \hookrightarrow \{1, 2\}$

GENERATED BY $w_1 \rightsquigarrow \mathbb{Z}[S] = M(1) \rightarrow W$

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}^2 \rightarrow \mathbb{Z}^3 \rightarrow \mathbb{Z}^4 \rightarrow \mathbb{Z}^5 \rightarrow \dots$$

EXERCISE: What's the kernel FI-module:

$$0 \rightarrow V \rightarrow M(1) \rightarrow W \rightarrow 0$$

FIND (MINIMAL?) GENERATORS FOR V .