Exercises

#1

$O_p(L) \leq H$. Hence $O_p(H) \leq O_p(L) \leq O_p(H)$

We want to show $Y_L = \Omega_1(Z(O_p(L)))$

Remember $L/O_p(L) = F^*(L/O_p(L))$

$Y_L \leq \Omega_1(Z(O_p(L)))$ since Y_L is p-reduced $O_p(L) \leq C(Y_L)$

We need to show $\Omega_1(Z(O_p(L)))$ is p-reduced.

<u>Claim:</u> $L/C_L(\Omega_1(Z(O_p(L))))$ has trivial O_p.

$O_p(L/C_L(\Omega_1(Z(O_p(L)))))$ is a normal section of $L/O_p(L)$

Let $R = L/O_p(L)$ then $R = F^*(R)$, $O_p(R) = 1$

so no proper normal section of R is a p-group.

#2

By (1) $Y_L = \Omega_1(Z(Q))$.

$1 \neq A \leq Z(Q) \Rightarrow N_G(A) \leq N_G(\Omega_1(A))$

may assume A is elementary abelian.

$L = C_L(Y_H) \overset{*}{=} C_L(Y_L) = C_L(\Omega_1(Z(O_p(H)))) \leq C_L(A)$. Then $L \leq N_G(A) \Rightarrow$ ✓

~~By lecture:~~ $\cancel{C_L(Y_L) = C_L(Y_H)}$ ~~[illegible]~~

* Lemma: H finite group, p a prime $L \trianglelefteq H$.

Then $C_L(Y_L) = C_L(Y_H)$.

KØBENHAVNS UNIVERSITET
UNIVERSITY OF COPENHAGEN

WWW.KU.DK

proof: Put $V = Y_L Y_H \trianglelefteq H$ $Y_L \leq O_p(L) \leq O_p(H)$ so
$[Y_L, Y_H] = 1$. Hence V is elementary abelian.
$Y_H = Y_V(H)$. Claim: $Y_L = Y_V(L) \cap L$
$Y_L \leq Y_V(L) \cap L$
Let $K \trianglelefteq L$ acting nilpotently on $Y_V(L) \cap L$.
$[Y_V(L), K] \leq [Y_V(L), L] \leq Y_V(L) \cap L$.
So K acts nilpotently on $Y_V(L)$
Hence K centralizes $Y_V(L)$ and so $Y_V(L) \cap L$
It follows that $Y_V \cap L$ is p-reduced
and $Y_V \cap L = Y_L$

And $C_L(Y_L) = C_L(Y_V(L) \cap L) = C_L(Y_V(L)) \overset{\text{By Lecture.}}{\leq} C_L(Y_V(H)) = C_L(Y_H)$ //

#3. $S \in Syl_p(P)$
(a) Suppose $C_P(Y_P) S = P \overset{Y_P\text{ - p-reduced}}{\Longrightarrow} C_P(Y_P) = P$
$\Rightarrow Y_P \leq Z(P) = 1$
Any minimal normal subgroup of P in $O_p(P)$ is
p-reduced, so $\cancel{Y}$.
(b) $O_p(P) \leq C_P(Y_P)$. Let $T \in Syl_p(C_P(Y_P))$. $T = S \cap C_P(Y_P)$
Frattini: $P = N_P(T) \cdot C_P(Y_P)$.
$P = C_P(Y_P) S \cdot N_P(T)$. $S \leq M \underset{max}{\leq} P$
$\Rightarrow C_P(Y_P) \cdot S \leq M$ $\Rightarrow N_P(T) \nleq M$ but $S \leq N_P(T)$
$\Rightarrow N_P(T) = P$. so $C_P(P) \leq T \leq O_p(P)$.

KØBENHAVNS UNIVERSITET
UNIVERSITY OF COPENHAGEN

WWW.KU.DK

b) We need to show $\Omega_1(Z(O_p(P)))$ is p-reduced.
Let $L \trianglelefteq P$ acting nilpotently on $\Omega_1(Z(O_p(P)))$ ↯
$LS = P$ or $LS \leq M$.
First case P acts nilpotently on $\Omega_1(Z(O_p(P)))$ so on V/p
and so $V/p \leq Z(P)$ ↯.
Second case : $LS \leq M \underset{max}{<} P$.
$P = L\,N_p(L \cap S) = L \cdot S\,N_p(L \cap S) \Rightarrow N_p(L \cap S) = P$
as in (c). $\Rightarrow L \cap S \trianglelefteq L$ $\quad L/O_p(L)$ is a p'-group.
$O_p(L) \leq O_p(P) \leq C_p(\Omega_1(Z(O_p(P))))$

#4 ~~Case~~ If $O_{p'}(G) \neq 1$.
Let $1 \neq g \in Z(Q)$ $\quad C_{O_{p'}(G)}(g) \overset{Q\text{ large}}{\leq} N_G(Q)$
$[C_{O_{p'}(G)}(g), Q] \leq O_{p'}(G) \cap Q = 1$
$\Rightarrow C_{O_{p'}(G)}(g) \leq O_{p'}(G) \cap C_G(Q) \leq O_{p'}(G) \cap Q = 1$
So $Z(Q)$ acts elementwise fixpointfreely
hence $Z(Q)$ is cyclic and by Thompson
~~since~~ $O_{p'}(G)$ is nilpotent.
Suppose $O_p(G) \neq 1$.
$A = C_{O_p(G)}(Q) \neq 1$ $\quad$ Let $K = O_{p'}(F(G))E(G) \leq C_G(O_p(G))$
$\leq C_G(A) \overset{\text{large}}{\leq} N_G(Q)$
$[K, Q] \leq K \cap Q \leq Z(K)$. $\Rightarrow [K, Q, K] = 1$.
$\Rightarrow [Q, O^p(K)] = 1 \Rightarrow [Q, K] = 1$. $\Rightarrow K \leq Q \Rightarrow K = O^p(K) = 1$

or $\quad K = F^*(K) \leq F^*(N_G(Q))$ and $Q \leq O_p(N_G(Q)) \Rightarrow [K, Q] = 1 \Rightarrow K = 1$.

This is 1.

From now on $O_p(G)=1$.

~~Let~~ Suppose G has a component K.

By K is not a p'-group. Let $L = \langle K^Q \rangle$

$Q \leq S \in Syl_p(G)$. $\quad L \cap S \neq 1$

$C_{L\cap S}(Q) \neq 1 \qquad L \cap Z(Q) \neq 1$.

Let $R = F(G)\langle X \mid X \in Comp(G),\ X \not\in L \rangle \Rightarrow [R, L] = 1$

$R \leq C_G(L\cap Q) \leq \mathcal{N}_G(Q)$ similar to

$R = 1$.

$Z(L) \leq F(G) \leq R = 1$. This is (3)

From now on Let $Comp(G) = 1$

$\Rightarrow E(G) = 1 \Rightarrow F^*(G) = F(G) \leq O_{p'}(G) \leq F(G) \leq F^*(G)$

so (2) is true.

#5 $\quad \hat{C} = N_G(\mathcal{U}) \qquad \begin{pmatrix} (\det(A))^{-1} & 0 \cdots 0 \\ * & \boxed{A} \end{pmatrix} \qquad O_p(\hat{C}) : \begin{pmatrix} 1 & 0 \cdots 0 \\ * & \boxed{I} \end{pmatrix}$

(a) G is doubly transitiv $\Rightarrow \hat{C}$ is maximal in G

since $\hat{C}$ is p-local and $O_p(G) = 1$, (a) holds.

(b) $Q \cong \mathbb{F}_q^{\,n-1}$

$\hat{C}/Q \cong \ ? \ SL_{n-1}(q) C_{q-1}$ so Q is p-reduced.

$\Rightarrow Q = Y_{\hat{C}}$.

(c) $C_G(Q) = ? \qquad C_{GL_n(q)}(Q) = Z(GL_n(q)) \times Q$

$C_G(Q) = Q$. $\quad E = O^p(F_p^*(C_G(Y_{\hat{C}}))) = O^p(F_p^*(Q)) = O^p(Q) = 1$.

(d) $1 \neq A \leq C_G(Q) \leq Q$. $\quad [V, A] = \mathcal{U}$