**Def:** Let $H$ be a finite group, $p$ a prime and $V$ an $\mathbb{F}_pH$-module.
(a) $V$ is $p$-reduced for $H$ if $O_p(H/C_H(V)) = 1$.
(b) $C_H^*(V)/C_H(V) = O_p(H/C_H(V))$.
(c) $H$ acts nilpotently (unipotently) if $[V, \underbrace{H, \dots, H}_{k}] = 1$ for some $k \in \mathbb{N}$.

**Remark:** (a) $V$ is $p$-reduced iff $C_H^*(V) = C_H(V)$.
(b) $H$ acts nilpotently on $V$ iff $H/C_H(V)$ is a $p$-group
iff $C_H^*(V) = H$.

**Lemma:** The following statements are equivalent:
(a) $V$ is $p$-reduced for $H$
(b) Any normal subgp. of $H$ which acts nilpotently on $V$ centralizes $V$.
(c) Any subnormal subgp. — " —

**Proof:** (a) $\Rightarrow$ (c): Let $L \trianglelefteq\trianglelefteq H$ acting nilpotently on $V$.
$\Rightarrow L/C_L(V)$ is a $p$-gp.
$\quad\; \shortparallel$
$LC_H(V)/C_H(V) \trianglelefteq\trianglelefteq H/C_H(V)$
$\Rightarrow LC_H(V)/C_H(V) \leq O_p(H/C_H(V))^{=1} \Rightarrow L \leq C_H(V)$.

(c) $\Rightarrow$ (b): ✓
(b) $\Rightarrow$ (a): $C_H^*(V)/C_H(V)$ is $p$-gp., hence acts nilpotently on $V$
$\Rightarrow C_H^*(V) = C_H(V) \Rightarrow V$ $p$-reduced. $\square$

**Lemma:** There exists a unique maximal $p$-reduced $H$-submodule $Y_V(H)$ of $V$.

**Proof:** $Y_V(H) := \langle W \mid W$ a $p$-reduced submodule of $V\rangle$

Claim: $Y_V(H)$ is $p$-reduced.

Let $L \trianglelefteq H$ act nilpotently on $Y_V(H)$ $\Rightarrow$ $L$ acts nilpotently on every p-reduced submodule $W$ of $V$

$\Rightarrow [W, L] = 1 \Rightarrow [Y_V(H), L] = 1.$ $\square$

Versions of the $P \times Q$ - Lemma:

<u>Lemma:</u> Let $P, Q \leq H$ s.t. $[P, Q] \leq C_H(V)$.

If $P$ acts nilpotently on $V$ and $V$ is p-reduced for $Q$ then
$$C_Q(V) = C_Q(C_V(P)).$$

<u>Proof:</u> Let $x \in C_Q(C_V(P))$ be a p'-elt.

$$V = C_V(x) \oplus [V, x]$$
$$C_{[V,x]}(P) \leq [V, x] \cap C_V(x) = 0$$

So as $P$ acts nilpotently, $[V, x] = 0$ $\Rightarrow x \in C_Q(V)$

$\Rightarrow C_Q(C_V(P))/C_Q(V)$ is a p-gp. and normal in $Q/C_Q(V)$

$\overset{Q \text{ p-reduced}}{\Longrightarrow}$ $C_Q(C_V(P)) = C_Q(V)$.

<u>Lemma:</u> Let $L \trianglelefteq\trianglelefteq H$.

(a) If $V$ is p-reduced for $H$ then $V$ is p-reduced for $L$.

(b) $Y_V(H) \leq Y_V(L)$

(c) $C_L(Y_V(L)) = C_L(Y_V(H))$

<u>Proof:</u> (a): $K \trianglelefteq\trianglelefteq L$ acting nilpotently on $V$ $\Rightarrow$ $K \trianglelefteq\trianglelefteq H$ and $K$ acts nilp. on $V$

~~An~~ Since $V$ p-reduced for $H$, $[V, K] = 1$.

(b): By (a), $Y_V(H)$ is p-reduced for $L$, so $Y_V(H) \leq Y_V(L)$.

(c): We may assume that $L \trianglelefteq H$. So $Y_V(L)$ is an $H$-submodule of $V$.

Let $W$ be an $H$-submodule of $Y_V(L)$ minimal with $C_L(W) = C_L(Y_V(L))$.

$O_p(L/C_L(W)) = O_p(L/C_L(Y_V(L))) = 1$.

Hence $W$ is p-reduced for $L$.

Let $K$ be normal in $H$ acting nilpotently on $W$.

$[L,K]$ is normal in $L$ and contained in $K$ acting nilpotently on $W$.

Since $W$ is $p$-reduced for $L$, $[L,K] \leq C_H(W)$

Can apply the previous lemma with $(L,K,W)$ in place of $(Q,P,V)$.

$$\Rightarrow \quad C_L(W) = C_L(C_W(K)) = C_L(Y_V(L))$$

So by minimality of $W$, $C_W(K) = W$, so $W$ is $p$-reduced for $L$.

$W \leq Y_V(H) \leq Y_V(L)$

$$\Rightarrow \quad C_L(Y_V(L)) \leq C_L(Y_V(H)) \leq C_L(W) \overset{\text{choice of } W}{=} C_L(Y_V(L))$$

$$\Rightarrow \quad C_L(Y_V(L)) = C_L(Y_V(H)).$$

<u>Lemma</u>: Let $S \in Syl_p(H)$ and $S \leq R \leq H$.

Let $U$ be a $p$-reduced $R$-submodule of $V$.

Then $\langle U^H \rangle$ is $p$-reduced for $H$.

In particular, $Y_V(R) \leq Y_V(H)$ and $C_V(S) = Y_V(S) \leq Y_V(H)$.

<u>Proof</u>: Put $W := \langle U^H \rangle$. Let $L \trianglelefteq H$ be acting nilpotently on $W$.

$R \cap L \trianglelefteq R$ is acting nilpotently on ($W$ and) $U$.

Since $U$ is $p$-reduced for $R$, $R \cap L \leq C_L(U)$

$L/C_L(W)$ is $p$-group. So $L = (L \cap S)\, C_L(W) = (L \cap R)\, C_L(W) \leq C_L(U)$

Since $L \trianglelefteq H$, $L$ centralizes $W = \langle U^H \rangle$.

<u>Recall</u>: If $H$ is a group then $Y_H$ is the largest $p$-reduced elementary abelian normal subgp.

<u>Lemma</u>: (a) $Y_H = Y_{\Omega_1 Z(O_p(H))}(H)$

(b) Let $L \trianglelefteq\trianglelefteq H$. Then $C_L(Y_L) = C_L(Y_H)$.

(c) Let $S \in Syl_p(H)$ and $S \leq R \leq H$ then $\Omega_1 Z(S) = Y_S \leq Y_R \leq Y_H$

$G$ $\mathcal{K}_p$-group of local characteristic $p$, i.e.

$$C_G(O_p(L)) = C_L(O_p(L)) \le O_p(L) \quad \text{for all } p\text{-local sgps. } L \text{ of } G$$

$S \in Syl_p(G)$

$\mathcal{L}(S) = \{ R \le G \mid S \le R,\ O_p(R) \ne 1 \}$

$\mathcal{M}(R)$ is the ~~a~~ set of maximal $p$-local subgroups of $G$ containing $R$.

<u>Lemma</u>: Let $R \in \mathcal{L}(S)$. Then $C_G(O_p(R)) \le O_p(R)$.

<u>Proof</u>: Set $L := N_G(O_p(R)) \Rightarrow L$ $p$-local

$O_p(L) \le S \in Syl_p(L)$

$O_p(L) \le R \Rightarrow O_p(L) \le O_p(R) \le O_p(L) \Rightarrow O_p(L) = O_p(R)$.

Let $G$ be a counterexample to the Small World Theorem.

$\tilde{C} \in \mathcal{M}(N_G(\Omega_1 Z(S)))$, $Q := O_p(\tilde{C})$, $Q \le L \Rightarrow L^\circ := \langle Q^L \rangle$.

$E := O^p(F_p^*(C_{\tilde{C}}(Y_{\tilde{C}})))$.

(1) $\mathcal{M}(S) \ne \{\tilde{C}\}$

(2) $\mathcal{M}(E) = \{\tilde{C}\}$

(3) $Y_R \le Q$ for all $R \in \mathcal{L}(S)$.

(4) There exists a unique $p$-minimal $P \in \mathcal{L}(S)$ with $P \not\le \tilde{C}$.

(5) $Y_P \le Q$ and $\langle Y_P^E \rangle$ is abelian.

(6) There exists a unique $p$-minimal $\tilde{P} \in \mathcal{L}(S)$ with $\tilde{P} \le ES$ and $\tilde{P} \not\le$ ~~$N_G(\tilde{P})$~~ $N_G(P^\circ)$

(7) Put $K = \langle O^p(\tilde{P})^{\tilde{C}} \rangle$. Then $K/O_p(K)$ is quasisimple.

Structure theorem ⇝ (8) Let $R \in \mathcal{L}(S)$ with $R \not\le \tilde{C}$. Then $[Y_R, R^\circ]$ is a natural $SL_2(q)$; $q \ge 2$ $Sp_{2n}(q)$, $n \ge 2$, ~~or~~ or $Sp_4(2)$-module. Moreover, in the $SL_n(q)$-case, $Y_R = [Y_R, R^\circ]$

(9) $Q$ is a large sgp. of $G$, i.e $C_G(Q) \le Q$ & $N_G(A) \le N_G(Q)$ for all $1 \ne A \le Z(Q)$

Put $L = N_G(P^\circ)$ and $H = K(L \cap \tilde{C})$.

**Lemma:** $L = P^\circ(H \cap L)$, $L^\circ = P^\circ$, $H = K(H \cap L) = E(H \cap L)$ and $H \cap L$ is a maximal subgp. of $H$ and of $L$.

**Proof:** $H \overset{H = K(L \cap \tilde{C})}{=} K(L \cap \tilde{C}) \leq \tilde{C} \Rightarrow H \cap L = \tilde{C} \cap L$

Frattini: $L = P^\circ N_L(P^\circ \cap S) = P^\circ N_L(Z(P^\circ \cap S))$

$Q \leq P^\circ$, $Q \leq P^\circ \cap S$ $\quad Z(P^\circ \cap S) \leq C_G(Q)$

As $Q$ is large $\Rightarrow N_G(Z(P^\circ \cap S)) \leq N_G(Q) = \tilde{C}$

$\Rightarrow L = P^\circ(L \cap \tilde{C}) = P^\circ(H \cap L)$

$L^\circ = \langle Q^L \rangle = \langle Q^{P^\circ} \rangle \overset{P = P^\circ S \text{ as } P \text{ p-minimal and } Q \not\leq O_p(P)}{=} \langle Q^P \rangle = P^\circ$ (If $Q \leq O_p(P)$ then $1 \neq A = Z(\langle Q^P \rangle) \leq C_G(Q)$, so $P \leq N_G(A) \leq N_G(Q) = \tilde{C}$ )

$H = K(L \cap \tilde{C}) = K(H \cap L)$, $K = \langle O^p(\tilde{P})^{\tilde{C}} \rangle$, $\tilde{P} \leq ES$, $O^p(\tilde{P}) \leq E$

$\Rightarrow K \leq E$

**Lemma:** $Y_P = Y_L$ and $Y_L$ is a natural $SL_2(q)$-module for $L^\circ$ (so $Y_P$ is a natural $SL_2(q)$-module for $P^\circ$ )

**Proof:** By (8), since $P$ is p-minimal, $Y_P$ is a natural $SL_2(q)$-module for $P^\circ$. Apply (8) to $L$ and $L^\circ$, use $Y_P \leq Y_L$ (as $S \leq P \leq L$) we get $Y_P = Y_L$.

( $R \in \mathcal{L}(S)$, $Q \not\leq R \Rightarrow Z(R^\circ) = 1$

Proof: As $Q$ is large, if $Z(R^\circ) \neq 1$ then $R \leq N(Z(R^\circ)) \leq N_G(Q)$ )

**Lemma:** $Y_H = C_{Y_P}(Q) = Y_{H \cap L}$ and $|Y_H| = q$

**Proof:** $P^\circ = \langle Q^P \rangle$, $P^\circ$ does not centralise $Y_P \Rightarrow C_{Y_P}(Q)$ is 1-dim. $\mathbb{F}_q$-subspace of the 2-dim. space $Y_P$.

$H \cap L$ normalizes $C_{Y_P}(Q) = C_{Y_L}(Q)$
($Q \trianglelefteq H \leq \tilde{C}$ )

$C_{Y_p}(Q)$ is an irreducible $H \cap L$-module

So $C_{Y_p}(Q) \leq Y_{H\cap L} \leq Y_H \cap Y_L$

$[Y_{H\cap L}, Q] = 1$ as $Q \leq O_p(H \cap L)$

$\Rightarrow C_{Y_p}(Q) = Y_{H\cap L}$

$H \leq \tilde{C} \Rightarrow Y_H \leq Y_{\tilde{C}}$, $E \leq C_{\tilde{C}}(Y_{\tilde{C}})$ by def. So $[Y_H, E] = 1$.

$H = E(H \cap L)$

$\Rightarrow Y_H$ is $p$-reduced for $H \cap L$

Hence $Y_H \leq Y_{H\cap L} \Rightarrow Y_H = Y_{H\cap L}$. $\square$

Set $G_0 = \langle H, L \rangle$

Lemma: (a) $O_p(G_0) = 1$.

(b) No non-trivial normal subgroup of $G_0$ is contained in $H \cap L$.

Proof: (a): Suppose $O_p(G_0) \neq 1$. Then $E \leq G_0 \leq N_G(O_p(G_0)) \leq \tilde{C}$ since $\mathcal{M}(E) = \{\tilde{C}\}$.
However, $P \leq L \leq G_0$ and $P \not\leq \tilde{C}$, contradiction.

(b): Let $A \leq H \cap L$ with $A \trianglelefteq G_0$.

$A \trianglelefteq H$. $F^*(A) \leq F^*(L)$ is a $p$-gp.

$\Rightarrow F^*(A)$ is a $p$-gp, $F^*(A) \trianglelefteq G_0 \Rightarrow F^*(A) = 1 \Rightarrow A = 1$. $\square$

Define a graph $\Gamma$.

Vertices of $\Gamma$: right cosets of $H$ and $L$ in $G_0$: $G_0/H \cup G_0/L$.

$A \in G_0/H$ is adjacent to $B \in G_0/L$ if $A \cap B \neq \emptyset$.

Notation: For $\alpha, \beta \in \Gamma$ define $G_\alpha = \{g \in G_0 \mid \alpha g = \alpha\}$
($\alpha = Hg$ then $G_\alpha = H^g$)

$Y_\alpha = Y_{G_\alpha}$

$\Delta(\alpha)$ - set of neighbours of $\alpha$ in $\Gamma$

$d(\alpha, \beta)$ = distance between $\alpha$ and $\beta$ in $\Gamma$.

If $\alpha = Hg$ then $E_\alpha = E^g$ and $\tilde{C}_\alpha = \tilde{C}^g$.

---

<u>Lemma</u>: (a) $G_0$ has two orbits on $\Gamma$, namely $G_0/H$ and $G_0/L$.
(b) $\Gamma$ is connected.
(c) $G_0$ acts transitively on the set of edges of $\Gamma$.
(d) $G_\alpha$ acts transitively on $\Delta(\alpha)$ for all $\alpha \in \Gamma$.
(e) $G_0$ acts faithfully on $\Gamma$.

<u>Proof</u>: (b): Let $\Gamma_0$ be the connected component of $\Gamma$ containing $\alpha = H$. Then $H \cap L \neq \emptyset$ so $\beta = L \in \Gamma_0$. $H$ fixes $\alpha$, so $H$ normalizes $\Gamma_0$, $L$ fixes $\beta$, so $L$ normalizes $\Gamma_0$.
$G_0 = \langle H, L \rangle \leq N_{G_0}(\Gamma_0)$. $\Rightarrow \Gamma_0 = \Gamma$ by (a).

(c): Let $\{Hg, Lk\}$ be an edge of $\Gamma$.
$\Rightarrow$ ex. $a \in Hg \cap Lk \Rightarrow Hg = Ha$, $Lk = La$, so $\{Hg, Lk\} = \{H, L\}a$.

(d): Let $\beta, \gamma \in \Delta(\alpha)$.
$\{\alpha, \beta\}g = \{\alpha, \gamma\}$ for some $g \in G_0$ by (c).
$\Rightarrow \alpha g = \alpha$ and $\beta g = \gamma \Rightarrow g \in G_\alpha$ and $\beta g = \gamma$.

(e): Let $A = C_{G_0}(\Gamma)$. Then $A \trianglelefteq G_0$, $A$ fixes $H$ and $L$.
So $A \leq H \cap L \Rightarrow A = 1$ by previous lemma.

<u>Lemma</u>: Let $\beta, \gamma \in G/H$ with $Y_\beta \cap Y_\gamma \neq 1$.
Then (a) $\tilde{C}_\beta = \tilde{C}_\gamma$, $E_\beta = E_\gamma$ and $Y_\beta = Y_\gamma$.
(b) Let $\delta \in \Delta(\gamma)$. Then $O_p(G_\beta) \cap O_p(G_\delta \cap G_\gamma) \leq O_p(G_\gamma)$.

<u>Proof</u>: (a): $S \leq H \leq \tilde{C}$ so $Y_H \leq Y_{\tilde{C}}$. $E$ centralizes $Y_H \Rightarrow \begin{cases} E_\beta \text{ centralizes } Y_\beta \\ E_\gamma \text{ —"— } Y_\gamma \end{cases}$

$\Rightarrow \langle E_\beta, E_\gamma \rangle \leq C_G(Y_\beta \cap Y_\gamma)$.

$\mathcal{M}(E) = \{\tilde{C}\}$, $\mathcal{M}(E_\beta) = \{\tilde{C}_\beta\}$, $\mathcal{M}(E_\gamma) = \{\tilde{C}_\gamma\}$
$\Rightarrow \{\tilde{C}_\beta\} = \mathcal{M}(E_\beta) = \mathcal{M}(C_G(Y_\beta \cap Y_\gamma)) = \mathcal{M}(E_\gamma) = \{\tilde{C}_\gamma\}$. So $\tilde{C}_\beta = \tilde{C}_\gamma$.

Let $g \in G_0$ with $\beta g = \gamma$. $g \in \tilde{C}_\beta$

$\tilde{C}_\beta^g = \tilde{C}_\gamma = \tilde{C}_\beta$, $E \trianglelefteq \tilde{C}$ and $Y_H \trianglelefteq \tilde{C}$ so $E_\beta \trianglelefteq \tilde{C}_\beta$ and $Y_\beta \trianglelefteq \tilde{C}_\beta$

$E_\gamma = E_\beta^g = E_\beta$

$Y_\gamma = Y_\beta^g = Y_\beta$. $\square$

(b):

<u>Claim</u>: $E_\beta$ normalizes $O_p(G_\beta) \cap O_p(G_\delta \cap G_\gamma)$.

$[O_p(G_\beta), E_\beta] \le O_p(G_\beta) \cap E_\beta \le O_p(E_\beta) = O_p(E_\gamma) \le O_p(G_\gamma) \le O_p(G_\delta \cap G_\gamma)$.

$\underset{\wedge |}{}$ $O_p(G_\beta) \cap O_p(G_\delta \cap G_\gamma)$. This proves the claim.

$H = E(H \cap L) \Rightarrow G_\gamma = E_\gamma(G_\gamma \cap G_\delta)$ (use here that $G_0$ is transitive on the edges)

$$O_p(G_\beta) \cap O_p(G_\delta \cap G_\gamma) \overset{E_\gamma = E_\delta}{\le} \bigcap_{g \in E_\gamma} O_p(G_\delta \cap G_\gamma)^g \le \bigcap_{g \in E_\gamma(G_\gamma \cap G_\delta)} O_p(G_\delta \cap G_\gamma)^g$$
$$= \bigcap_{g \in G_\gamma} O_p(G_\delta \cap G_\gamma)^g \le O_p(G_\gamma).$$

$\square$

<u>Def</u>: $b = \min\{d(\alpha, \alpha') \mid \alpha, \alpha' \in \Gamma, Y_\alpha \nleq O_p(G_{\alpha'})\}$.

$\left( \bigcap_{\alpha' \in \Gamma} O_p(G_{\alpha'}) \le O_p(G_0) = 1. \right)$

Pick $\alpha, \alpha' \in \Gamma$ with $Y_\alpha \nleq O_p(G_{\alpha'})$ and $d(\alpha, \alpha') = b$.

Let $\alpha, \alpha+1, \alpha+2, \ldots, \alpha'-1, \alpha'$ be a shortest path from $\alpha$ to $\alpha'$.

<u>Lemma</u>: (a) $\alpha \in G_0/L$ and $\alpha+1 \in G_0/H$.

(b) $b \ge 2$.

<u>Proof</u>: $Y_\alpha$ is a normal $p$-sgp. of $G_\alpha$. So $Y_\alpha \le O_p(G_\alpha) \Rightarrow \alpha \ne \alpha'$, $b \ge 1$.

(a): Suppose $\alpha \in G_0/H \Rightarrow \alpha+1 \in G_0/L$. We may assume $\alpha = H$ and $\alpha+1 = L$.

$$Y_\alpha = Y_H = Y_{H \cap L} \le Y_L = Y_{\alpha+1} \underset{\uparrow \text{ minimality of } b}{\le} O_p(G_{\alpha'}) \quad \lightning \; Y_\alpha \nleq O_p(G_{\alpha'}).$$

(b): Assume $\alpha = L$ and $\alpha+1 = H$ (may do that by (a)).

$Y_\alpha = Y_L \overset{\text{Property (3) in Setup}}{\le} Q \le O_p(H) = O_p(G_{\alpha+1}) \Rightarrow \alpha+1 \ne \alpha' \Rightarrow b \ge 2$.

Lemma: $b$ is odd.

Proof: Suppose $b$ is even. So $\alpha \in G_0/L$ and $\alpha' \in G_0/L \Rightarrow \alpha'-1 \in G_0/H$.

We may assume $\alpha'-1 = H$ and $\alpha' = L$.

$$Y_\alpha \leq O_p(G_{\alpha'-1}) = O_p(H) \leq O_p(H \cap L) \leq S \leq P$$

$$Y_{\alpha'} = Y_L = Y_P$$

By one of the exercises, $O_p(P) \in Syl_p(C_P(Y_P))$.

Claim: $[Y_\alpha, Y_{\alpha'}] \neq 1$.

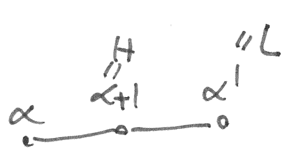
Suppose $[Y_\alpha, Y_{\alpha'}] = 1 \Rightarrow Y_\alpha \leq C_P(Y_P) \Rightarrow Y_\alpha \leq O_p(P) \cap O_p(H \cap L)$

$$L = P(H \cap L)$$

$$Y_\alpha \leq \bigcap_{g \in P} O_p(H \cap L)^g = \bigcap_{g \in P(H \cap L)} O_p(H \cap L) \leq O_p(L) = O_p(G_{\alpha'}) \quad ↯.$$

So $[Y_\alpha, Y_{\alpha'}] \neq 1$. Since $Y_\alpha$ is p-reduced, ~~so~~ $Y_{\alpha'} \not\leq O_p(G_\alpha)$.

Claim: $b > 2$.




Suppose $b = 2$.

Since $G_{\alpha+1}$ is transitive on $\Delta(\alpha+1)$, $\alpha = \alpha'^g$, $g \in G_{\alpha+1} = H$.

$[Y_\alpha, Y_{\alpha'}] \neq 1$. $Y_{\alpha'} = Y_L = Y_P$, $Y_\alpha = Y_{\alpha'}^g = Y_P^g$

$\Rightarrow [Y_P, Y_P^g] \neq 1$.

$g \in H = E(H \cap L)$, $H \cap L$ normalizes $Y_P = Y_L$.

So $Y_P^g = Y_P^e$ for some $e \in E$.

$[Y_P, Y_P^e] \neq 1$, a contradiction since $\langle Y_P^E \rangle$ is abelian by (5).

$Y_{\alpha+1} \leq O_p(G_{\alpha'}) \leq C_{G_{\alpha'}}(Y_{\alpha'})$. So $[Y_{\alpha+1}, Y_{\alpha'}] = 1$.

$[Y_\alpha, Y_{\alpha'}] \neq 1$. $Y_\alpha$ is a natural $SL_2(q)$-module for $G_\alpha^{\circ}$ ($Y_L$ is a nat. $SL_2(q)$-module for $L^{\circ}$)

$\Rightarrow [Y_\alpha, Y_{\alpha'}] = Y_{\alpha+1}$

By symmetry, $[Y_{\alpha'}, Y_\alpha] = Y_{\alpha'-1}$ so $Y_{\alpha+1} = Y_{\alpha'-1}$.

$\delta \quad \alpha-1 \quad \alpha \quad \alpha+1 \quad \cdots \quad \alpha'-1 \quad \alpha'$

Let $\alpha-1 \in \Delta(\alpha)$, $\delta \in \Delta(\alpha-1)$ $\quad (\alpha-1 \neq \alpha+1)$

$b > 2 \Rightarrow b > 3$ as $b$ is even.

$\Rightarrow Y_\delta \leq O_p(G_{\alpha+1}) \Rightarrow [Y_\delta, Y_{\alpha+1}] = 1 \overset{Y_{\alpha+1} = Y_{\alpha'-1}}{\Rightarrow} [Y_\delta, Y_{\alpha'-1}] = 1$.

$Y_\delta \leq O_p(G_{\alpha'-3}) \Rightarrow [Y_\delta, Y_{\alpha'-3}] = 1$

Note $N_G(Y_H) = \tilde{C}$, $N_L(Y_H) = H \cap L$ so 1-1 correspondence between elts. of $\Delta(\alpha)$ and 1-dim. subspaces of $Y_\alpha$.

$\Rightarrow Y_{\alpha'-2} = Y_{\alpha'-3} Y_{\alpha'-1}$.

$\Rightarrow [Y_\delta, Y_{\alpha'-2}] = 1 \Rightarrow Y_\delta \leq O_p(G_{\alpha'-2}) \leq O_p(G_{\alpha'-1} \cap G_{\alpha'-2})$.

$Y_\delta \leq O_p(G_{\alpha'-1} \cap G_{\alpha'-2}) \cap O_p(G_{\alpha+1}) \leq O_p(G_{\alpha'-1})$ by earlier lemma.

$Y_{\alpha'-1} \cap Y_{\alpha+1} = Y_{\alpha'-1} = Y_{\alpha+1} \neq 1$

So $Y_\delta \leq O_p(G_{\alpha'-1}) \leq G_{\alpha'}$

$[Y_\delta, Y_{\alpha'-1}] = 1 \Rightarrow [Y_\delta, Y_{\alpha'}] \leq Y_{\alpha'-1} = Y_{\alpha+1} \leq Y_\alpha$

$V_{\alpha-1} = \langle Y_\delta \mid \delta \in \Delta(\alpha-1) \rangle \Rightarrow [V_{\alpha-1}, Y_{\alpha'}] \leq Y_\alpha \leq V_{\alpha-1}$

$V_{\alpha-1} \trianglelefteq G_\alpha \cap G_{\alpha-1}$, $V_{\alpha-1} \trianglelefteq G_{\alpha-1} \Rightarrow V_{\alpha-1} \ntrianglelefteq G_\alpha$ since $G_0 = \langle H, L \rangle = \langle G_\alpha, G_{\alpha-1} \rangle$.

$G_\alpha \cap G_{\alpha-1}$ max. sgp. of $G_\alpha$, so $Y_{\alpha'} \leq N_{G_\alpha}(V_{\alpha-1}) = G_\alpha \cap G_{\alpha-1}$

$\Rightarrow Y_{\alpha'}$ normalizes $Y_{\alpha-1}$, $[Y_\alpha, Y_{\alpha'}] = Y_{\alpha+1}$

$Y_{\alpha-1} \cap Y_{\alpha+1} = 1 \Rightarrow [Y_{\alpha-1}, Y_{\alpha'}] = 1$

$Y_\alpha = Y_{\alpha-1} Y_{\alpha+1} \Rightarrow [Y_\alpha, Y_{\alpha'}] = 1$ $\quad \lightning$

How to do the $b$ odd case:

$\beta = \alpha + 1$

determine $V_\beta$ as a module for $K_\beta$

$SL$

$Sp_{2n}(q)$