Partial Groups

$\mathcal{L} \neq \emptyset$, $\mathbb{W} = \mathbb{W}(\mathcal{L})$ = free monoid on $\mathcal{L}$

$w = (g_1, \dots, g_n)$

(1) $\mathbb{D} \subseteq \mathbb{W}$ with $\mathcal{L} \subseteq \mathbb{D}$, and
$u \circ v \in \mathbb{D} \Rightarrow u, v \in \mathbb{D}$ $\quad (- \text{so } (\emptyset) \in \mathbb{W})$

(2) $\Pi : \mathbb{D} \to \mathcal{L}$, $\Pi|_{\mathcal{L}} = id$
$\Pi(u \circ v \circ w) = \Pi(u \circ (\Pi(v)) \circ w)$

(3) (Inversion) $g \mapsto g^{-1}$, $w \mapsto (g_n^{-1}, \dots, g_1^{-1})$
$u \in \mathbb{D} \Rightarrow u^{-1} \circ u \in \mathbb{D}$ and $\Pi(u^{-1} \circ u) = 1 \overset{\text{def}}{=} \Pi(\emptyset)$.

$\mathcal{H} \leq \mathcal{L}$ (partial subgroup)

$\emptyset \neq \mathcal{H} = \mathcal{H}^{-1}$

$w \in \mathbb{W}(\mathcal{H}) \cap \mathbb{D} \Rightarrow \Pi(w) \in \mathcal{H}$

$\beta : \mathcal{L} \to \mathcal{L}'$ (homomorphism)

$\beta^* : \mathbb{W} \to \mathbb{W}'$

$\mathbb{D}\beta^* \subseteq \mathbb{D}'$, $\Pi'(w\beta^*) = (\Pi(w))\beta$.

Examples:

1. Take a centric linking system $\mathcal{L}^c$ a la [BLO]
$Iso(\mathcal{L}^c)/\approx$, $\varphi \approx \psi$ if φ is a "restriction or extension of ψ" .

2. G finite group, $S \in Syl_p(G)$, $\emptyset \neq \Delta \subseteq Subgps(S)$
Take $\mathbb{D} = \{ w = (g_1, \dots, g_n) \in \mathbb{W}(G) \mid \exists\ (P_0, \dots, P_n) \in \mathbb{W}(\Delta)$ with
$P_0 \xrightarrow{c_{g_1}} P_1 \to \dots \xrightarrow{c_{g_n}} P_n \}$

Π = multivariable product in G.

<u>Objective partial group</u> $(\mathcal{L}, \Delta)$, $\mathcal{L}$ a partial group

$\Delta \subseteq \mathrm{Subgps}(\mathcal{L})$ s.t. ("Δ set of objects")

(O1) $\mathbb{D}$ is as in Example (2)

(O2) Any subgroup of an object containing a conjugate of an object is an object.

<u>Localities</u>: A <u>Locality</u> is a triple $(\mathcal{L}, \Delta, S)$ s.t. $\mathcal{L}$ is finite, $(\mathcal{L}, \Delta)$ an objective partial group, S = unique maximal member of Δ (via $\subseteq$), S maximal in the poset of p-subgroups of $\mathcal{L}$.

"transporter system"

Δ-<u>linking system</u>: locality $(\mathcal{L}, \Delta, S)$ s.t. $\mathcal{F}_S(\mathcal{L})$ is saturated,

$\mathcal{F}^{cr} \subseteq \Delta \subseteq \mathcal{F}^q$ ($P \in \mathcal{F}^q$ means $\exists\, Q \in P^{\mathcal{F}} \cap \mathcal{F}^f$ s.t. $C_{\mathcal{F}}(P) = \mathcal{F}_{C_S(P)}(C_S(P))$.)

and $C_{\mathcal{L}}(P)$ is a p-sgp. of $\mathcal{L}$ for all $P \in \Delta$.

$$C_{\mathcal{L}}(P) = \{g \in \mathcal{L} \mid (g^{-1}, x, g) \in \mathbb{D} \text{ and } \Pi(g^{-1}, x, g) = x \ \ \forall x \in P\}$$

"Theory" of localities: quotients by "partial normal subgroups", 1st iso theorem, ...

<u>Assume</u> henceforth $(\mathcal{L}, \Delta, S)$ is a Δ-linking system.

$\mathcal{N} \trianglelefteq \mathcal{L}$ if $\mathcal{N} \leq \mathcal{L}$ <u>and</u> s.t. $(g^{-1}, x, g) \in \mathbb{D}$ with $x \in \mathcal{N}$ $(g \in \mathcal{L}) \Rightarrow x^g \in \mathcal{N}$.

Partial subgp. $\langle X \rangle$ of $\mathcal{L}$ generated by $X \subseteq \mathcal{L}$.

$$\langle X \rangle \overset{\text{def}^n}{=} \bigcap \{\mathcal{H} \leq \mathcal{L} \mid X \subseteq \mathcal{H}\}$$

<u>Thm A1</u>. (Thm B in [BCGLO1])

Let $\mathcal{F}^{cr} \subseteq \Delta \subseteq \Delta^+ \subseteq \mathcal{F}^q$, $\mathcal{F}$ saturated.

Δ, Δ^+ closed w.r.to overgroups in S and $\mathcal{F}$-conjugation, and let $(\mathcal{L}, \Delta, S)$ be a Δ-linking system over $\mathcal{F}$.

Then $\exists!\ (\mathcal{L}^+, \Delta^+, S)$ Δ^+-linking system whose <u>restriction</u> to Δ is $\mathcal{L}$.

Moreover $\mathcal{L}^+$ is over $\mathcal{F}$.

Thm A2: $(\mathcal{L}, \mathcal{L}^+)$ as in Thm A1. Let $\mathcal{N} \trianglelefteq \mathcal{L}$ and set
$\mathcal{N}^+ = \langle \mathcal{N}^{\mathcal{L}^+} \rangle \leq \mathcal{L}^+$. Then $\mathcal{N}^+ \trianglelefteq \mathcal{L}^+$ and $\mathcal{N}^+ \cap \mathcal{L} = \mathcal{N}$.
Moreover, there is a bijection

$$(\trianglelefteq \mathcal{L}) \underset{- \cap \mathcal{L}}{\overset{\langle -^{\mathcal{L}^+} \rangle}{\rightleftarrows}} (\trianglelefteq \mathcal{L}^+)$$

Next theme: Fix $\mathcal{N} \trianglelefteq \mathcal{L}$, $(\mathcal{L}, \Delta, S)$ a Δ-linking system on $\mathcal{F}$ (with $\Delta = \mathcal{F}^q$).
Set $T := S \cap \mathcal{N}$, $\mathcal{E} = \mathcal{F}_T(\mathcal{N})$ (fusion system on T generated by conjugations $U \xrightarrow{c_g} T$, $U \leq T$, $g \in \mathcal{N}$).

Defn: $\mathcal{N}$ is large (in $\mathcal{L}$) if $C_S(\mathcal{N}) \leq T$.

Theorem B: (with $\Delta = \mathcal{F}^q$) Suppose $\mathcal{N}$ is large. Then $\mathcal{E}^c \subseteq \mathcal{F}^q$.

More generally ($\mathcal{N}$ not necessarily large); set $\Sigma = \{ C_S(\mathcal{N}) U \mid U \in \mathcal{E}^c \}$.
Find $\Sigma \subseteq \mathcal{F}^q (= \Delta)$.

Let $C_{\mathcal{L}}(\mathcal{N}) = \bigcap \{ C_{\mathcal{L}}(N_{\mathcal{N}}(P)) \mid P \in \Sigma \} \leq \mathcal{L}$.
and set $O^p_{\mathcal{L}}(\mathcal{N}) = \langle O^p(N_{\mathcal{N}}(P)) \mid P \in \Sigma \rangle$.

Proposition: $O^p_{\mathcal{L}}(\mathcal{N}) \trianglelefteq \mathcal{L}$, $O^p_{\mathcal{L}}(\mathcal{N}) T = \mathcal{N}$, and $O^p_{\mathcal{L}}(\mathcal{N})$ is minimal subject to these conditions.

Write $\mathcal{L}_T$ for $N_{\mathcal{L}}(T)$ ($\leq \mathcal{L}$ not necessarily a subgp. of $\mathcal{L}$)
$(\mathcal{L}_T, \Delta, S)$ is a Δ-linking system.
Write $\mathcal{C}_T$ for $C_{\mathcal{L}}(T)$. Get $\mathcal{C}_T \trianglelefteq \mathcal{L}_T$, and so $O^p_{\mathcal{L}_T}(\mathcal{C}_T) \trianglelefteq \mathcal{L}_T$.

Theorem C ($\Delta = \mathcal{F}^q$)

(a) $C_{\mathcal{L}}(\mathcal{N}) = O^p_{\mathcal{L}_T}(\mathcal{C}_T) C_S(\mathcal{N})$

(b) $C_{\mathcal{N}}(\mathcal{L}) \trianglelefteq \mathcal{L}$ and $C_{\mathcal{L}}(\mathcal{N}) \mathcal{N} \trianglelefteq \mathcal{L}$.

(c) Let $x \in C_{\mathcal{L}}(\mathcal{N})$, $y \in \mathcal{N}$ with $(x, y) \in \mathbb{D}$. Then $(y, x) \in \mathbb{D}$ and $xy = yx$.
(Implies $(x^{-1}, y, x) \in \mathbb{D} \Rightarrow (y^{-1}, x, y) \in \mathbb{D}$ & $y^x = y$, $x^y = x$)

Thm D: ($\Delta = \mathcal{F}^q$)

Set $\Delta_0 = \{P \in \Delta \mid P \cap T \in \mathcal{E}^c\}$ ($\supseteq \mathcal{E}^c$ by theorems B & C)

and set $\mathcal{N}_0 = \{g \in \mathcal{N} \mid c_g$ sends members of Δ_0 into $S\}$

(a) $(\mathcal{N}_0, \mathcal{E}^c, T)$ is an $\mathcal{E}^c$ - linking system on $\mathcal{E}$ (and "so" $\mathcal{E}$ is saturated.)

(b) $\langle \mathcal{N}_0 \rangle = \mathcal{N}$.

(c) $\mathcal{N}_0 \xhookrightarrow{\text{incl}} \mathcal{L}$ is a hom. of partial gps inducing a simplicial map

$$\mathbb{D}(\mathcal{N}_0) \to \mathbb{D}(\mathcal{L})$$

Remark: (Broto and Gonzalez) This is a ~~weak~~ homotopy equivalence

$$|\mathcal{L}_{[BLO]}| \simeq \mathbb{D}(\mathcal{L})$$

Theorem E: ($\Delta = \mathcal{F}^q$)

$$\{\trianglelefteq \mathcal{L}\} \xrightarrow{\text{bij}} \{\trianglelefteq \mathcal{F}\}$$
$$\mathcal{N} \longmapsto \mathcal{F}_T(\mathcal{N})$$

? introduces the next theme: considering more than one $\mathcal{N}$.

Def^n: ($\Delta = \mathcal{F}^q$?)

$$F^*(\mathcal{L}) = \bigcap \{\mathcal{N} \trianglelefteq \mathcal{L} \mid O_p(\mathcal{L}) \leq \mathcal{N} \ \& \ C_S(\mathcal{N}) \leq \mathcal{N}\}$$

Prop: $F^*(\mathcal{L})$ is the smallest large partial normal sgp. of $\mathcal{L}$ containing $O_p(\mathcal{L})$.

Def^n: Set $\mathcal{M} = F^*(\mathcal{L})$ (where $\Delta = \mathcal{F}^q$).

Define Δ_0 as in thm D (with $\mathcal{M}$ in place of $\mathcal{N}$).

$(\mathcal{M}_0, \mathcal{E}^c, T)$ linking system

$\mathcal{E} = \mathcal{F}_T(\mathcal{M})$, $T = S \cap \mathcal{M}$.

Def^n: A component of $\mathcal{L}|_{\Delta_0}$ is a partial normal subgp. $\mathcal{K} \trianglelefteq \mathcal{M}_0$ s.t. $\mathcal{K}$ is not a p-gp., and minimal for these conditions.

Proposition ($\Delta = \mathcal{F}^q$) $\mathcal{M}_0 = O_p(\mathcal{L}) E(\mathcal{L}_0)$ where $E(\mathcal{L}_0)$ is the product of the components of $\mathcal{L}_0$ and where distinct components commute elementwise.

Moreover: $\mathcal{L}_0 = N_{\mathcal{L}}(T)\mathcal{M}_0$ ($T \in \Delta_0$, $N_{\mathcal{L}}(T)$ is a sgp. of $\mathcal{L}_0$).