Solutions to the Exercises

Exercise 1. By 1.2, $L \leq L(G)$, so as $L(G) = E(G)$ by hypothesis, we have $L \leq E(G)$. Therefore L acts on each component of G and centralizes $F(G)$ by 1.1.2. Hence by 1.1.3, there exists a component D of G not centralized by L. Set $D_0 = DD^t$; we may assume $G = D_0 L\langle t\rangle$. In particular, $[G, t] \leq D_0$.

If t centralizes D then D is a component of $C_G(t)$, so by 1.1 and as L does not centralize D, we have $L = D$, so that (1) holds in this case. Thus we may assume t does not centralizes D, so $D_0 = [G, t]$. Now $F^*(G) = D_0 X$, where $X = C_{F^*(G)}(D_0)$. As $D_0 = [G, t]$, $[X, t] \leq Z(X) = Z$, so $Z_0 = Z\langle t\rangle \trianglelefteq X_0 = X\langle t\rangle$. Thus if K is a component of X then $[K, Z_0] = 1$ (cf. 31.4 in [FGT]), so K is a component of $C_G(t)$. Hence by 1.1, either $K = L$ or K centralizes L. In the first case (1) holds, so we may assume L centralizes $E(X)$. We've seen that L centralizes $F(G) = F(X)$, so $L \leq C_{E(G)}(X) = D_0$. Thus if $D_0 = D$ then (2) holds, so we may assume $D \neq D^t$. Set $Y = \{dd^t : d \in D\}$ and observe the map $\varphi : d \mapsto dd^t$ is a homomorphism of D onto Y, so as D is quasisimple, so is Y. Further $C_{D_0}(t) = Y C_{Z(D_0)}(t)$ so $Y = E(C_{D_0}(t))$ and then as L is a component of $C_G(t)$, we conclude that $L = Y$, so that (3) holds.

Exercise 2. First we prove the two parts of Lemma 5.4.

(1) Let $S_0 = \ker(\alpha)$. First ϕ^α is a well defined injection, since for $x, y \in S$ we have $x\alpha = y\alpha$ iff $xy^{-1} \in S_0$. Then as S_0 is strongly closed in S and ϕ is a monomorphism, $xy^{-1} \in S_0$ iff $(xy^{-1})\phi \in S_0$ iff $x\phi \in S_0(y\phi)$ iff $x\phi\alpha = y\phi\alpha$.

Also $\phi^\alpha : P\alpha \to Q\alpha$ is a homomorphism: $((xy)\alpha)\phi^\alpha = (xy)\phi\alpha = (x\phi \cdot y\phi)\alpha = x\phi\alpha \cdot y\phi\alpha = (x\alpha)\phi^\alpha \cdot (y\alpha)\phi^\alpha$.

(2) Let $\phi \in \hom_{\mathcal{F}}(P, Q)$ and $x \in \ker(\alpha_{|P})$. Then $x\alpha = 1$, so $(x\alpha)\phi^\alpha = 1\phi^\alpha = 1$. But $(x\alpha)\phi^\alpha = x\phi\alpha$, so $x\phi \in \ker(\alpha)$. As this holds for all ϕ and $x \in \ker(\alpha_{|P})$, it follows from the definition of strong closure that $\ker(\alpha)$ is strongly closed in S with respect to $\mathcal{F}$.

Now we prove Lemma 7.1. Here $\theta : S \to S^+ = S/S_0$ is the natural map $\theta : x \mapsto x^+ = xS_0$. For $P, Q \leq S$, $P^+ = P\theta$ and $\phi^+ : P^+ \to Q^+$ defined by $\phi^+ : x^+ \mapsto (x\phi)^+$ is the map ϕ^θ from Lemma 5.4. Thus by 5.4.1, $\phi^+ : P^+ \to Q^+$ is a well defined injective homomorphism.

As $S_0 \trianglelefteq \mathcal{F}$, ϕ extends to $\varphi \in \hom_{\mathcal{F}}(PS_0, QS_0)$; observe that $\phi^+ = \varphi^+$ as a map from $P^+ = (PS_0)^+$ to $Q^+ = (QS_0)^+$. Thus we may always choose P, Q to be the full preimage of P^+ and Q^+ in S; that is choose S_0 to be contained in P and Q.

Now if $\phi^+ \in \hom_{\mathcal{F}^+}(P^+, Q^+)$ and $\psi^+ \in \hom_{\mathcal{F}^+}(Q^+, R^+)$, then as S_0 is contained in P, Q, R, $\phi \in \hom_{\mathcal{F}}(P, Q)$ and $\psi \in \hom_{\mathcal{F}}(Q, R)$, so as $\mathcal{F}$ is a category, $\phi\psi \in \hom_{\mathcal{F}}(P, R)$. Then for $x \in P$, $x^+(\phi\psi)^+ = (x\phi\psi)^+ = ((x\phi)\psi)^+$, while $x^+(\phi^+\psi^+) = ((x^+)\phi^+)\psi^+ = ((x\phi)^+)\psi^+ = ((x\phi)\psi)^+$. Therefore $(\phi\psi)^+ = \phi^+\psi^+$. Also if id is the identity map on P, then $(id)^+$ is the identity map on P^+, so $\mathcal{F}^+$ is indeed a category.

Let $s \in S$ with $P^s \leq Q$. Then as $\mathcal{F}$ is a fusion system, $c_s : P \to Q$ is in $\hom_{\mathcal{F}}(P, Q)$, while for $x \in P$, $(x^+)(c_s)^+ = (xc_s)^+ = (x^s)^+ = (x^+)^{s^+} = (x^+)(c_{s^+})$, so $c_{s^+} \in \hom_{\mathcal{F}^+}(P^+, Q^+)$.

Finally as $\mathcal{F}$ is a fusion system, $\phi : P \to P\phi$ and its inverse $\phi^{-1} : P\phi \to P$ are $\mathcal{F}$-maps. Then $\phi^+ : P^+ \to P^+\phi^+$ is a $\mathcal{F}^+$-map and for $y^+ \in P^+\phi^+$, $(y^+)(\phi^{-1})^+ = (y\phi^{-1})^+$, so $(\phi^{-1})^+ = (\phi^+)^{-1}$. Thus $\mathcal{F}^+$ is a fusion system.

By definition of $\mathcal{F}^+$, $\phi^\theta = \phi^+ \in \hom_{\mathcal{F}^+}(P^+, Q^+)$, so from section 5, $\theta : \mathcal{F} \to \mathcal{F}^+$ is a morphism of fusion systems. Thus it remains to show that θ is surjective. By construction, $\theta : S \to S^+$ is surjective, and by definition, for each $\phi^+ \in \hom_{\mathcal{F}^+}(P^+, Q^+)$, $\phi^+ = \phi^\theta$, completing the proof.

Exercise 3. As S is quaternion of order $m \geq 16$, $S = \langle x, y\rangle$ where $|y| = m/2$, $|x| = 4$, $y^x = y^{-1}$ and $x^2 = z$ is the involution in $Y = \langle y\rangle$. The set $\mathcal{S}$ of subgroups of S consists of the set $\mathcal{Y}$ of subgroup of Y, the set $\mathcal{X}$ of subgroups $\langle xy^i\rangle$, each of order 4 and squaring to z, and the set $\mathcal{Q}$ of subgroups $\langle y^i, xy^j\rangle$, such that $y^i \notin Z = \langle z\rangle$, each of which is quaternion of order $2|y^i|$. As the automorphism group of a cyclic 2-group is a 2-group, $Aut(T)$ is a 2-group for $T \in \mathcal{Y} \cup \mathcal{X}$. Similarly if $T = \langle y^i, x\rangle \in \mathcal{Q}$ and $|y^i| > 4$ then $Y_T = \langle y^i\rangle$ is the unique cyclic subgroup of T of index 2, and hence characteristic in T. But then if α is an automorphism of T of odd order then α is trivial on Y_T and T/Y_T, so α is trivial. Thus the set $\mathcal{Q}_8$ of Q_8-subgroups of S is the set of subgroups Q such that $Aut(Q)$ is *not* a 2-group; indeed $Aut(Q) \cong S_4$.

Suppose $\mathcal{F}$ is a saturated fusion system on S. By Alperin's Fusion Theorem, $\mathcal{F} = \langle Aut_{\mathcal{F}}(R) : R \in \mathcal{F}^{frc}\rangle$. Let $R \in \mathcal{F}^{frc}$. If $Aut(R)$ is a 2-group then $Aut_{\mathcal{F}}(R)$ is a 2-group. But as R is radical, $Inn(R) = O_2(Aut_{\mathcal{F}}(R)) = Aut_{\mathcal{F}}(R)$, so $N_S(R) = RC_S(R) = R$, as R is centric. Therefore $R = S$ if $Aut_{\mathcal{F}}(R)$ is a 2-group. Thus we have shown:

(a) $\mathcal{F}^{frc} = \{S\} \cup \mathcal{R}$, where $\mathcal{R} = \{R \in \mathcal{Q}_8 : Aut_{\mathcal{F}}(R) \text{ is not a 2-group}\}$.

Indeed as $m > 8$, for $Q \in \mathcal{Q}_8$, $Aut_S(R) \cong D_8$, so as $Aut(Q) \cong S_4$, $Q \in \mathcal{R}$ iff $Aut_{\mathcal{F}}(Q) = Aut(Q)$. Then by Alperin's Fusion Theorem:

(b) $\mathcal{F} = \langle Aut_S(S), Aut(R) : R \in \mathcal{R}\rangle$.

Thus by (b), $\mathcal{F} = S$, $\mathcal{F}_1$, $\mathcal{F}_2$, or $\mathcal{F}_{1,2}$. To show each of these systems $\mathcal{F}$ is saturated, it suffices to show there exists a finite group $G(\mathcal{F})$ with Sylow 2-subgroup S such that $\mathcal{F} = \mathcal{F}_S(G(\mathcal{F}))$.

By definition, $S = \mathcal{F}_S(S)$; that is $G(S) = S$ and S is saturated.

Let $G = SL_2(q)$ with $(q^2-1)_2 = m$. Then $S \in Syl_2(G)$ and for each $Q \in \mathcal{Q}_8$, $O^2(N_G(Q)) \cong SL_2(3)$, so $Q \in \mathcal{R}$. Therefore $\mathcal{F}_S(G) = \mathcal{F}_{1,2}$; that is $G(\mathcal{F}_{1,2}) = G$.

Similarly suppose $q = q_0^2$ is a square. Then the centralizer H_0 of a involutory field automorphism of G is isomorphic to $SL_2(q_0)$ and we may choose notation so that S acts on H_0. Set $H = H_0S$, so that $|H : H_0| = 2$ and $H_0 = O^2(H)$. We may choose notation so that $Q_1 \leq H_0$; thus from the previous paragraph, $O^2(N_{H_0}(Q_1)) \cong SL_2(3)$. On the other hand $Q_2 \not\leq H_0$, so $O^2(N_H(Q_2)) = 1$. Thus setting $\mathcal{E} = \mathcal{F}_S(H)$, $\mathcal{R}(\mathcal{E}) = Q_1^S$, so $\mathcal{E} = \mathcal{F}_1$; that is $H = G(\mathcal{F}_1)$.

Let M be a quaternion group of order $2m$ with $S \leq M$. Then for some $y \in M - S$, $Q_1^y = Q_2$, so if α is conjugation by y then $\alpha \in Aut(S)$ with $Q_1\alpha = Q_2$, so

$$\mathcal{F}_1\alpha^* = \langle Aut_S(S), Aut(Q_1)\rangle\alpha^* = \langle Aut_S(S)\alpha^*, Aut(Q_1)\alpha^*\rangle = \langle Aut_S(S), Aut(Q_1)\rangle = \mathcal{F}_2.$$

Exercise 4. By assumption in Lemma 6.3, Q is strongly closed in S. Thus by Lemma 6.2 it suffices to show that Q is contained in each member of $\mathcal{F}^{frc}$.

Let $R \in \mathcal{F}^{frc}$; as $Q_1 \trianglelefteq \mathcal{F}$, $Q_1 \le R$ by 6.2. By 6.4 there is a model G for $N_{\mathcal{F}}(R)$ and $R = O_p(G)$. Thus $T = N_S(R) \in Syl_p(G)$ and $C_G(R) \le R$. Set $P = N_Q(R)$, so that $P \le T$. Set $P_R = R \cap P$. As Q is strongly closed in S, $P_R \trianglelefteq G$ and $[P, R] \le P_R$.

As $Q_1 \le R$ and $Q_1 \trianglelefteq \mathcal{F}$, $Q_1 \trianglelefteq G$. Therefore $H = C_G(Q_1) \trianglelefteq G$. By assumption in 6.3.2, $Aut_{\mathcal{F}}(Q_1)$ is a p-group, so $Aut_G(Q_1)$ is a p-group. Therefore $O^p(G) \le H$.

By assumption in 6.3.2, Q/Q_1 is abelian, so $[P, P_R] \le Q_1$. Therefore $[H, P] \le C_H(R/P_R) \cap C_H(P_R/Q_1)$, so as H centralizes Q_1, as Q_1 and P_R are normal in G, and as $C_G(R) \le R$, it follows that $[H, P] \le R$. Then as $O^p(G) \le H$, $[O^p(G), P] \le R$, so $P \le O_p(G) = R$. Therefore $Q = P \le R$, completing the proof.

Exercise 5. (1) implies (2): Let $1 \ne X$ be a 2-subgroup of K and $g \in N_G(X)$, Then $1 \ne X = X^g \le K \cap K^g$, so as K is tightly embedded, $K = K^g$. Thus $N_G(X) \le N_G(K)$. Let $h \in G$ with $X^h \le K$. Then $1 \ne X^h \le K \cap K^h$, so $K = K^h$. Thus $h \in N_G(K)$, so $X^G \cap K = X^{N_G(K)}$.

(2) implies (3): Trivial.

(3) implies (1): Assume $g \in G$ and $|K \cap K^g|$ is even. Thus there is an involution $x \in K$ with $x^g \in K$. By (3) there is $h \in N_G(K)$ with $x^{gh} = x$. Then again by (3), $gh \in C_G(x) \le N_G(K)$, so $g = (gh)h^{-1} \in N_G(K)$. Hence $K = K^g$, so that (1) holds.

Exercise 6. As $T \in \mathcal{F}^f$ and $\mathcal{F}$ is saturated, each $\phi \in Aut_{\mathcal{F}}(T)$ extends to some $\hat{\phi} \in \Sigma$. Thus $Aut_{\mathcal{F}}(T) = Aut_{\Sigma}(T)$, proving (1). As $Q_0 = C_S(T)$ and Σ acts on T and TQ_0, Σ acts on Q_0, proving (2).

Let $X \in \tilde{\mathcal{X}}$. We first observe that if $\sigma \in \hom_{\mathcal{F}}(XT, S)$ then $X\sigma \in \tilde{\mathcal{X}}(\mathcal{C}\sigma^*)$. Let $\alpha \in \mathfrak{A}(X)$ and $Y = X\alpha$. As $X \in \tilde{X}$, $\mathcal{C}\alpha^*$ is a component of $\mathcal{N} = N_{\mathcal{F}}(Y)$. By 5.2.2 there is $\beta \in \mathfrak{A}(X\sigma)$ with $X\sigma\beta = Y$. Let $\zeta = \alpha^{-1}\sigma\beta \in \hom_{\mathcal{F}}((XT)\alpha, S)$. Then $Y\zeta = X\sigma\beta = Y$, so $Y \in \hom_{\mathcal{N}}(T\alpha, N_S(Y))$, and hence as $\mathcal{C}\alpha^*$ is a component of $\mathcal{N}$, so is $\mathcal{C}\alpha^*\zeta^* = \mathcal{C}\sigma^*\beta^*$. Therefore $X\sigma \in \tilde{\mathcal{X}}(\mathcal{C}\sigma^*)$.

Next assume $\sigma \in \Sigma$ with $X\sigma \in \tilde{\mathcal{X}}$. We've seen that $\mathcal{C}\sigma^*\beta^*$ is a component of $\mathcal{N}$. But also as $X\sigma \in \tilde{\mathcal{X}}$, $\mathcal{D} = \mathcal{C}\beta^*$ is a component of $\mathcal{N}$. Now $T\beta$ is Sylow in $\mathcal{D}$ and as $\sigma \in \Sigma$, $T\sigma = T$, so $T\sigma\beta = T\beta$ is Sylow in $\mathcal{C}\sigma^*\beta^*$. By 10.1, $E(\mathcal{N})$ is the central product of its components, so either $\mathcal{D} = \mathcal{C}\sigma^*\beta^*$ or the components are distict and commute. However in the latter case, $T\beta$ centralizes $T\beta$, so $T\beta$ is abelian and hence normal in $\mathcal{D}$ by 6.4.1, contradicting $\mathcal{D}$ quasisimple. Therefore $\mathcal{C}\beta^* = \mathcal{D} = \mathcal{C}\sigma^*\beta^*$, so $\mathcal{C} = \mathcal{C}\sigma^*$ and hence (3) holds.

Suppose X is characteristic in Q_0. Then Σ acts on X by (2). Thus by (3), $Aut_{\Sigma}(T) \le Aut(\mathcal{C})$, so (4) follows from (1).

Finally we prove (5). By (1) it suffices to show for each $\sigma \in \Sigma$, $\sigma_{|T} \in Aut(\mathcal{C})$. We show that $1 \ne Q \cap Q\sigma$; then (5) follows from (3). Suppose that $1 = Q \cap Q\sigma$. Then as $QQ\sigma \le Q_0$, $|Q|^2 = |QQ\sigma| \le |Q_0|$, contrary to the hypotheses of (5).

Exercise 7. By definition of a split extension, Q is a complement to S_0 in S, so $S = S_0Q$ and $S_0 \cap Q = 1$. As Q is weakly closed in S and S_0 is strongly closed in S, $[S_0, Q] \le S_0 \cap Q = 1$. That is

(a) $S = S_0 \times Q$.

For $P \leq S$ set $P_0 = P \cap S_0$. Let $R \in \mathcal{F}^{frc}$; by 6.4 there is a model G for $N_{\mathcal{F}}(R)$ with $R = O_2(G)$. Let $T \in Syl_2(G)$ and $H = O^2(G)$. As $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ and $S = S_0 Q$, $O^2(\mathcal{F}) = O^2(\mathcal{F}_0)$, so $T \cap H \leq T_0$. Therefore $C_H(R_0) = Z(R_0)$. Let $P = N_Q(R)$. Then by (a), P centralizes R_0, so $[P, H] \leq C_H(R_0) = Z(R_0)$, so P centralizes H. Therefore as $P \trianglelefteq T$, P is normal in $HT = G$. Therefore $P \leq R$, so as $P = N_Q(R)$, we conclude $Q = P \trianglelefteq G$. Hence Q is $Aut_{\mathcal{F}}(R)$-invariant, so by Alperin's Fusion Theorem, Q is strongly closed in S with respect to $\mathcal{F}$ and hence by 6.2:

(b) $Q \trianglelefteq \mathcal{F}$.

Further $T = Q \times T_0$ and $H \cap Q \leq T_0 \cap Q = 1$, so $G = HT_0 \times Q$. Therefore by Alperin's Fusion Theorem:

(c) For each $X \leq Q$ and $\psi \in \hom_{\mathcal{F}}(X, S)$, $\psi = c_{y|X}$ for some $y \in Q$.

Finally let $P_0 \leq S_0$ and $\phi \in \hom_{\mathcal{F}_0}(P_0, S_0)$. By (b), ϕ lifts to $\varphi \in \hom_{\mathcal{F}}(P_0 Q, S)$ acting on Q. By (c), $\varphi = \phi \times c_y$ for some $y \in Q$, so $\mathcal{F} = \mathcal{F}_0 \times Q$ by definition of the direct product in section 8.