2-FUSION SYSTEMS OF COMPONENT TYPE

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Introduction. This series of lectures involves the interplay between local group theory and the theory of fusion systems, with the focus of interest the possibility of using fusion systems to simplify part of the proof of the theorem classifying the finite simple groups.

For our purposes, the classification of the finite simple groups begins with the Gorenstein-Walter Dichotomy Theorem (cf. [ALSS]) which says that each finite group G of 2-rank at least 3 is either of component type or of characteristic 2-type. This supplies a partition of the finite groups into groups of odd and even characteristic, from the point of view of their 2-local structure. We will be concerned almost exclusively with the groups of odd characteristic: the groups of component type. However Ulrich Meierfrankenfeld's lectures can be thought of as being concerned with the groups of even characteristic.

In the case of a saturated fusion system $\mathcal{F}$, the situation vis-a-vis the Gorenstein-Walter dichotomy is nicer: $\mathcal{F}$ is either of characteristic p-type or component type, irrespective of rank. Further the Dichotomy Theorem for saturated fusion systems is much easier to prove than the theorem for groups; indeed once the notion of the *generalized Fitting subsystem* $F^*(\mathcal{F})$ of a saturated fusion system $\mathcal{F}$ is put in place, and suitable properties of $F^*(\mathcal{F})$ are established, including E-balance, the proof of the Dichotomy Theorem for fusion systems is easy.

But of more importance, it seems easier to work with 2-fusion systems of component type than with groups of component type. This is because in a group G of component type, a 2-local subgroup H of G may have a nontrivial *core*, where the core of H is the largest normal subgroup $O(H)$ of H of odd order. The existence of these cores introduces big problems into the analysis of groups of component type. These problems can be minimized if one can prove the B-Conjecture, which says that, in a simple group,

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cores of 2-locals are “small” in a suitable sense. Unfortunately the proof of the B-Conjecture is quite difficult. However in a saturated fusion system there are no cores in local subsystems. Thus the corresponding difficulties do no arise, and no effort need be expended on the B-Conjecture. This is the major reason why one can hope that a classification of simple saturated 2-fusion systems of component type should be easier than the classification of simple groups of component type.

There is a program in progress that attempts, first, to translate some of the mathematics used to classify the finite simple groups of component type into the category of saturated 2-fusion systems, and then, second, to classify the saturated simple 2-fusion systems of component type using that mathematics. Finally, third, one can hope to use the classification of the fusion systems of component type to obtain a classification of the finite simple groups of component type. Hopefully this would provide a simplification of the proof of the theorem classifying the finite simple groups. In particular such an approach would avoid the necessity of classifying the unbalanced groups, which was the basis for the proof of the B-Conjecture.

I will begin my series of lectures with a discussion of groups of component type, including an outline of the classification of such groups. Then I will provide some background on fusion systems and the translation of local finite group theory into a local theory of fusion systems. Finally I’ll outline a program for classifying simple saturated 2-fusion systems of component type, indicating on the one hand which parts of the program are complete or nearly complete, and, on the other, where important notions from local group theory have not as yet been translated into analogous notions for fusion systems, or the necessary theorems on fusion systems have not as yet been proved.

Section 1. The generalized Fitting subgroup of a finite group.

Let G be a finite group. Our reference for basic finite group theory is [FGT].

Recall G is *quasisimple* if $G = [G, G]$ and $G/Z(G)$ is simple. Subnormality is the transitive extension of the normality relation on subgroups of G. The *components* of G are the subnormal quasisimple subgroups of G.

Let $E(G)$ be the subgroup of generated by the components of G. Recall $F(G)$ is the *Fitting subgroup* of G, which is the largest normal nilpotent subgroup of G. Thus $F(G)$ is the direct product over all prime divisors of $|G|$ of the groups $O_p(G)$, where $O_p(G)$ is the largest normal p-subgroup of G. Finally $F^*(G) = F(G)E(G)$ is the *generalized Fitting subgroup* of G. We record the following standard facts, which can be found for example in section 31 of [FGT].

Lemma 1.1. *(1) $E(G)$ is the central product of the components of G, which are permuted by G via conjugation.*

(2) $F^*(G)$ is the central product of $F(G)$ and $E(G)$.

(3) $C_G(F^*(G)) = Z(F^*(G))$.

Recall next that $O(G) = O_{2'}(G)$ is the largest normal subgroup of G of odd order. Write $O_{2',E}(G)$ for the preimage in G of $E(G/O(G))$ and set $L(G) = O^{2'}(O_{2,E}(G))$. The subgroup $L(G)$ is the *layer* of G.

Lemma 1.2. *(L-balance) For each 2-local subgroup H of G, $L(H) \leq L(G)$.*

Proof. See 31.17 in [FGT] for a proof using the Schreier Conjecture.

Lemma 1.3. *Let t be an involution in G and L a component of $C_G(t)$. Suppose $L(G) = E(G)$. Then there exists a component D of G such that one of the following holds:*

(1) $L = D$.

(2) $L < D = [D, t]$.

(3) $D \neq D^t$ and $L = E(C_{DD^t}(t)) = \{dd^t : d \in D\}$ is the image of D under the map $d \mapsto dd^t$.

Proof. This is Exercise 1.

Hypothesis 1.4. Let $\mathcal{I}$ be the set of involutions in G and assume for each $t \in \mathcal{I}$ that $E(C_G(t)) = L(C_G(t))$. Write $\mathcal{L}$ for the set of components of centralizers of involutions in G. For $L \in \mathcal{L}$, write $[L]$ for the set of $K \in \mathcal{L}$ such that L is a homomorphic image of K, and set $[\mathcal{L}] = \{[L] : L \in \mathcal{L}\}$.

Lemma 1.5. *Assume Hypothesis 1.4. Assume $t, s \in \mathcal{I}$, L is a component of $C_G(t)$, and s centralizes t and L. Then there exists a component D of $C_G(s)$ such that one of the following holds:*

(1) $L = D$.

(2) $L < D = [D, t]$.

(3) $D \neq D^t$, $L = E(C_{DD^t}(t))$, and $D \in [L]$.

Proof. Set $H = C_G(s)$, and observe that as L is a component of $C_G(t)$ centralized by s, L is also a component of $C_H(t)$. By Hypothesis 1.4, $E(H) = L(H)$, so the lemma follows from 1.3 applied to H in the role of G.

Notation 1.6. Assume Hypothesis 1.4. Define $[L_0] \in [\mathcal{L}]$ to be *maximal* if whenever $L \in [L_0]$ and $t, s \in \mathcal{I}$ with L a component of $C_G(t)$ such that s centralizes t and L, then case (2) of Lemma 1.5 is *not* satisfied. For example if L_0 is of maximal order in $\mathcal{L}$ then $[L_0]$ is maximal. Later we will see that the Component Theorem says that if G satisfies Hypothesis 1.4 and $[L]$ is maximal, then, modulo an exceptional case, L is a so-called *standard subgroup* of G. This is accomplished using some combinatorics arising from the tension generated by Lemma 1.5.

Our finite group G is said to be *almost simple* if $F^*(G)$ is a nonabelian simple group. In that event, setting $L = F^*(G)$, the conjugation map $c : G \to Aut(L)$ is (by 1.1.3) an injection embedding G in $Aut(L)$ with $Lc = Inn(L)$. Thus an almost simple group is a subgroup of $Aut(L)$ containing $Inn(L)$ for some nonabelian simple group L.

Here is one of a number of (roughly) equivalent statements of the B-conjecture:

B-Conjecture. If G is almost simple then for each involution t in G, $L(C_G(t)) = E(C_G(t))$.

Put another way, the B-conjecture says that almost simple groups satisfy Hypothesis 1.4. The B-conjecture was derived as a corollary to the Unbalanced Group Theorem:

Unbalanced Group Theorem. *Assume G is an almost simple group possessing an involution t such that $O(C_G(t)) \neq 1$. Then $F^*(G)$ is on a list of known simple groups.*

Section 2. The simple groups and notions of characteristic.

Here is one statement of the Classification Theorem:

Classification Theorem. *Each finite simple group is isomorphic to one of the following:*

(1) A group of prime order.

(2) The alternating group A_n on a set of order n.

(3) A finite group of Lie type.

(4) One of 26 sporadic groups.

The groups of Lie type in (3) are linear groups over finite fields. Thus if G is of Lie type then as a linear group it comes equipped with a characteristic: the characteristic of the defining finite field.

The local group theory underlying the classification focuses on the 2-local subgroups of finite groups G, so we wish to come up with a definition of the notion of a group of "even characteristic" in terms of the 2-local structure of the group, such that, from the point of view of this definition, the groups of Lie type over fields of even order are of even characteristic, and those over fields of odd order are of odd characteristic.

We begin with the definitions used in the original proof of the classification.

Definition 2.1. A finite group G is of *characteristic 2-type* if for each 2-local subgroup H or G, $F^*(H) = O_2(H)$. Further G is of *component type* if for some 2-local H, $L(H) \neq 1$. Given a prime p, the *p-rank* $m_p(G)$ of G is the maximum of the dimensions of elementary abelian p-subgroups of G, viewed as vector spaces over the field of order p.

In the original proof of the classification, the groups of characteristic 2-type are regarded as groups of even characteristic, and those of component type are regarded as of odd characteristic. The groups G with $m_2(G) \leq 2$ are regarded as "small". Then the Dichotomy Theorem says that, if G is not too small, then G is of even or odd characteristic.

Gorenstein-Walter Dichotomy Theorem. *Let G be a finite group with $m_2(G) \geq 3$. Then G is either of characteristic 2-type or of component type.*

The proof of the Dichotomy Theorem uses L-balance, but also the Feit-Thompson Theorem on groups of odd order, signalizer functor theory, and the Bender-Suzuki classification of groups with a strongly embedded subgroup.

The simple groups of Lie type over fields of even order are of characteristic 2-type, while most groups of Lie type over fields of odd order are of component type, as are most alternating groups. Roughly half of the sporadic groups are of component type, and of course the rest are of characteristic 2-type.

The Dichotomy Theorem supplies a partition of the simple groups into groups of even and odd characteristic. But it is not clear that partition is optimal. Here are two other notions of "even groups".

Definition 2.2. Define G to be of *even characteristic* if for each 2-local H of G containing a Sylow 2-subgroup S of G, $F^*(H) = O_2(H)$. Define the *Thompson group* $J(S)$ of S to be the subgroup generated by the elementary abelian 2-subgroups of S of 2-rank $m_2(S)$. The *Baumann subgroup* of S is $\mathrm{Baum}(S) = C_S(\Omega_1(Z(S)))$. We say G is of *Baumann characteristic 2* if for each 2-local subgroup H of G containing $\mathrm{Baum}(S)$, we have $F^*(H) = O_2(H)$.

The following two examples begin to suggest that it might be advantageous to move the boundary of the odd-even partition so as to make more groups of even characteristic.

Example 2.3. Let $X = X(q^2)$ be a group of Lie type over a field $\mathbf{F}_{q^2}$ of even order and t an involutory field automorphism of X. Form the semidirect product $G = X\langle t\rangle$. Then (generically) $F^*(C_X(t)) = L$ is of Lie type $X(q)$, so that G is of component type. But, aside from some small G, each 2-local H of G containing a Sylow 2-subgroup S of G, or even just with $m_2(H) = m_2(G)$, satisfies $F^*(H) = O_2(H)$. That is G is of even characteristic and of Baumann characteristic 2. This seems more satisfying, since intuitively G *should* be an "even group".

Example 2.4. Let $X = X_1 \times X_2$ be the direct product of two copies X_i of $X(q)$ with q even, let t be an involutory automorphism of X with $X_1^t = X_2$, and form $G = X\langle t\rangle$. This time $L = C_X(t) = \{xx^t : x \in X_1\} \cong X_1$, so G is of component type. Indeed for s an involution in X_2, X_1 is a component of $C_G(s)$. But for each 2-local H containing a Sylow 2-subgroup S, $F^*(H) = O_2(H)$, so G is of even characteristic. If $s \in Z(S \cap X_2)$ then s centralizes $\mathrm{Baum}(S)$, so G is not of Baumann characteristic 2. But that is not so bad. The point is that it is difficult to work with $C_G(t)$ as its 2-share is so small. But $|S : C_S(s)| = 2$, so $C_G(s)$ is much easier to work with, and to use to recognize that X_1 is a component of G.

If one adopts either of the definitions of "even groups" in Definition 2.2, then difficulties encounter in dealing with involution centralizers like those in Examples 2.3 and 2.4 are avoided or minimized, as the groups in 2.3 become "even", while in 2.4 the 2-share of the order of the centralizer of the relevant involution s is large. On the other hand the class of "even groups" is enlarged, so the treatment of such groups becomes more difficult. But perhaps the trade is a good one.

The Gorenstein-Lyons-Solomon program has yet another definition of "even group" which also has the effect of enlarging the class. The Aschbacher-Smith treatment of quasithin groups is devoted to groups of even characteristic.

Finally here is one more possible definition of an "even group": G is even if $F^*(H) = O_2(H)$ for each 2-local subgroup H of G with $m_2(H) = m_2(G)$. This notion of an "even group" sits somewhere between that of groups of Baumann characteristic 2 and groups of characteristic 2-type.

Section 3. The Component Theorem.

As usual let G be a finite group.

Definition 3.1. A subgroup K of G is said to be *tightly embedded* in G if K is of even order and the intersection of each pair of distinct conjugates of K is of odd order. A *standard subgroup* of G is a quasisimple subgroup L such that $K = C_G(L)$ is tightly embedded in G, $N_G(K) = N_G(L)$, and L commutes with none of its conjugates.

A $\mathcal{K}$-group is a finite group G with the property that each simple section of G is on the list $\mathcal{K}$ of "known" simple groups appearing in the statement of the Classification Theorem. A $\hat{\mathcal{K}}$-group is a finite group each of whose proper subgroups is a $\mathcal{K}$-group. In particular a minimal counter example to the Classification Theorem is a $\hat{\mathcal{K}}$-group.

There are a variety of results on tightly embedded subgroups in the literature. In essence they say that if $Q \in Syl_2(K)$, $m_2(K) > 1$, and $\Phi(Q) \neq 1$, then, aside from a few exceptional Q, K is subnormal in G.

The situation when L is standard with centralizer K is even more restrictive; here either $L \trianglelefteq G$ or $m_2(K) = 1$ or Q is a 4-subgroup.

Example 3.2. Let $G \cong A_n$ with $n \geq 9$ be the alternating group on $I = \{1, \dots, n\}$, let Q be the 4-subgroup of G moving $J = \{1, 2, 3, 4\}$, and $L = G_J \cong A_{n-1}$. Then L is a standard subgroup of G with Q Sylow in $K = G_{I-J} \cong A_4$.

Or let V be an n-dimesional symplectic space over $\mathbf{F}_q$ with q odd and $n \geq 6$, $G = PSp(V)$, U a nondegenerate 2-dimensional subspace of V, $L = C_G(U)$, and $K = C_G(U^\perp)$. Then L is standard in G with centralizer $K \cong SL_2(q)$ of 2-rank 1.

Component Theorem. *Assume Hypothesis 1.4 and $L \in \mathcal{L}$ with $[L]$ maximal. Then one of the following holds:*

(1) L is standard in G.

(2) $m_2(L) = 1$ and $L_0 = \langle L^G \cap N_G(L) \rangle = L * L^g$ with $C_G(L_0)$ tightly embedded in G.

(3) There exist $D \in [L]$ and commuting involutions t and s such that L is a component of $C_G(t)$, D is a component of $C_G(s)$, $D \neq D^t$ appears in case (2), and $L = E(C_{DD^t}(t))$.

(4) L is a component of G or $L = E(C_{DD^t}(t))$ for some component $D \neq D^t$ of G and involution t.

In particular if G is almost simple then case (4) of the Component Theorem does not hold, while in case (3) we can replace L by D to reduce to case (2). Hence, in an almost simple group G satisfying Hypothesis 1.4, either G has a standard subgroup or case (2) arises. As in Example 3.2, the group $G = PSp_4(q)$ provides an example where case (2) holds.

Richard Foote [F] determined all groups in which case (2) of the Component Theorem holds. Alternatively, one can use the Classical Involution Theorem from the next section to treat that case.

Observe that if G is almost simple of component type and the B-Conjecture holds, then we can choose L to satisfy the hypotheses of the Component Theorem. Thus by the Component Theorem and the discussion in the previous paragraph, we can assume that L is a standard subgroup of G. This leads to:

Standard Form Problem for L. Let L be a known quasisimple group. Determine the finite groups G with a standard subgroup isomorphic to L.

From the discussion above, if we can prove the B-Conjecture and solve the standard form problem for each known quasisimple group L, then we will have determined all almost simple $\hat{\mathcal{K}}$-groups of component type.

In practice it is not necessary to do the standard form problem for L when L is of Lie type and odd characteristic and not $L_2(q)$ or a Ree group. Instead one uses Walter's Theorem, as discussed in the next section. Thus we need only consider the cases where L is $L_2(q)$ or a Ree group, or a covering of an alternation group, a group of Lie type in characteristic 2, or a sporadic group.

But Examples 2.3 and 2.4 show that the case where L is of Lie type and characteristic 2 can be difficult. Thus it might be preferable to assume G is not of even characteristic or Baumann characteristic 2, so that there exists a 2-central involution t or an involution t in the center of the Baumann subgroup such that $C_G(t)$ has a component. Then one would need to refine the Component Theorem to show there exists a standard subgroup L such that $C_G(L)$ contains such an involution. In our work on quasithin groups, Steve

Smith and I were able to do this in quasithin group not of even characteristic. Magaard and Stroth have a preprint which does something similar in general.

However I have no good ideas as to how to prove a Baumann Component Theorem. I do have an approach in mind to deal with the fusion theoretic analogue of the situtation where $L(C_G(t)) \neq 1$ for some involution t in G with $m_2(C_G(t)) = m_2(G)$.

Section 4. The Classical Involution Theorem and Walter's Theorem.

As usual let G be a finite group.

Definition 4.1. A *classical involution* in G is an involution z such that $C_G(z)$ has a subnormal subgroup K such that $K/O(K) \cong SL_2(q)$ or $\hat{A}_7$ for some odd prime power q, and $z \in K$. Here $\hat{A}_7$ is the covering of A_7 with quaternion Sylow 2-groups.

Example 4.2. Given an odd prime p, let $Chev^*(p)$ consist of the quasisimple groups G of Lie type over a field of characteristic p other than $L_2(q)$ and the Ree groups. If $G = X(q)$ is such a group a *fundamental subgroup* of G is a subgroup $K = \langle U, U^- \rangle$, where U is the center of a long root subgroup of G and U^- is an opposite of U. It turns out that $K \cong SL_2(q)$ and K is subnormal in $C_G(z)$, where z is the involution in K. If G is $G_2(q)$ or ${}^3D_4(q)$ we can also take U to be a short root subgroup, in which case $K \cong SL_2(q)$ or $SL_2(q^3)$ in the respective case. Thus we also regard these SL_2-subgroups as fundamental subgroups. In particular in each case, z is a classical involution.

Classical Involution Theorem. *Assume z is a classical involution in G with $O(G) = 1$ and $z \in K$ with $K/O(K) \cong SL_2(p^e)$ or $\hat{A}_7$ and K is subnormal in $C_G(z)$. Assume G is the subnormal closure of K in G. Then one of the following holds:*

(1) $G \in Chev(p)^*$ and K is a fundamental subgroup.
(2) $G = K$ is $SL_2(3)$ or $\hat{A}_7$.
(3) $G \cong M_{11}$.

Actually something much stronger is proved under a fusion theoretic hypothesis which includes the classicial involution setup as a special case, and also includes the case where G has a tightly embedded subgroup with quaternion Sylow 2-subgroups. We will encounter the fusion theoretic hypothesis in the third lecture.

The Classical Involution Theorem supplies a characterization of the groups in $Chev^*(p)$. In [W] John Walter used this characterization to prove the following theorem:

Walter's Theorem. *Assume G is almost simple such that each subgroup of G which does not contain $F^*(G)$ is a $\mathcal{K}$-group. Assume there exists an involution t in G and L subnormal in $C_G(t)$ with $L/O(L) \in Chev^*(p)$. Then $G \in Chev^*(p)$.*

Actually when $p = 3$ a few more groups are excluded as choices for $L/O(L)$, but this statement suffices for expository purposes. Harris proves a similar theorem in [H].

As discussed in the previous section, Walter's Theorem allows us to avoid the standard form problems for groups in $Chev^*(p)$. Walter also uses his theorem (and various other difficult theorems) to determine the list of "unbalanced groups" appearing as conclusions in the Unbalanced Group Theorem from section 1. Hence his theorem leads to a proof of the B-conjecture.

Given the B-conjecture and the Component Theorem, the classification of $\hat{\mathcal{K}}$-groups of component type is then reduced to the solution of the standard form problems for the list of quasisimple groups appearing near the end of Section 3. Recall that a Sylow 2-group Q of $K = C_G(L)$ is either of 2-rank 1 or a 4-group. Using the classification of groups with a tightly embedded subgroup with quaternion Sylow 2-groups, we can assume Q is cyclic or a 4-group. Thus for t an involution in K, $C_G(t)$ closely resembles the centralizer in some almost simple group. It is then left to show G is isomorphic to that group, using the structure of the centralizer.

Section 5. Fusion systems.

Our basic references for fusion systems are [AKO], [BLO], and [Cr].

Let p be a prime. A *fusion system* on a finite p-group S is a category whose objects are the subgroups of S and such that the set $\hom_{\mathcal{F}}(P,Q)$ of morphisms from a subgroup P of S to a subgroup Q of S consists of injective group homomorphisms; in addition two weak axioms are required to hold. See Definition I.2.1 in [AKO] for a precise definition.

The Standard Example. Let G be a finite group and S a Sylow p-subgroup of G. For $g \in G$, write c_g for the conjugation map $c_g : x \mapsto x^g = g^{-1}xg$ on G. Let $\mathcal{F}_S(G)$ be the fusion system on S where $\hom_{\mathcal{F}_S(G)}(P,Q)$ consists of the maps $c_g : P \to Q$ for those $g \in G$ with $P^g \leq Q$. We will call $\mathcal{F}_S(G)$ the *p-fusion system of* G.

Let $\mathcal{F}$ be a fusion system on S. We will call S the *Sylow group* of $\mathcal{F}$. The fusion system $\mathcal{F}$ is said to be *saturated* if it satisfies two additional axioms, which are easily verified in the Standard Example using Sylow's Theorem. There are various choices for the two additional axioms; see Definition I.2.2 and Proposition I.2.5 in [AKO] for two possible sets of axioms.

Fusion systems and saturated fusions systems were first defined by Puig, beginning about 1990. However Puig didn't publish his work for about 15 years, and in the interim others took up his work and introduced their own terminology. For example Puig calls a fusion system on a p-group S a *divisible S-category* and he calls saturated fusion systems *Frobenius categories*. However here we will use the (by now) standard terminology of "fusion system" and "saturated fusion system". See Puig's book [P] for some history and for Puig's approach to the subject.

For $P \leq S$ set $P^{\mathcal{F}} = \{P\phi : \phi \in \hom_{\mathcal{F}}(P,S)\}$. The members of $P^{\mathcal{F}}$ are the $\mathcal{F}$-*conjugates* of P.

Definition 5.1. A subgroup P of S is said to be *fully normalized*, *centric* if for each $Q \in P^{\mathcal{F}}$, $|N_S(P)| \geq |N_S(Q)|$, $C_S(Q) \leq Q$, respectively. The subgroup P is *radical* if $O_p(Aut_{\mathcal{F}}(P)) = Inn(P)$. Write $\mathcal{F}^f$, $\mathcal{F}^c$, $\mathcal{F}^r$ for the fully normalized, centric, radical subgroups of S, respectively. We also write $\mathcal{F}^{frc}$ for $\mathcal{F}^f \cap \mathcal{F}^r \cap \mathcal{F}^c$, and use other similar notation.

Let $\mathfrak{A}(P) = \mathfrak{A}_{\mathcal{F}}(P)$ consists of those $\alpha \in \hom_{\mathcal{F}}(N_S(P), S)$ such that $P\alpha \in \mathcal{F}^f$.

Lemma 5.2. *Let $P \leq S$.*

(1) If $P \in \mathcal{F}^f$ then $Aut_S(P) \in Syl_p(Aut_{\mathcal{F}}(P))$.

(2) For each fully normalized conjugate Q of P, there exists $\alpha \in \mathfrak{A}(P)$ with $P\alpha = Q$.

Proof. See for example Proposition I.2.5 and Lemma I.2.6 in [AKO].

Given a subgroup T of S and subcategories $\mathcal{T}_i$, $i \in I$, of $\mathcal{F}$ on subgroups of T, write $\langle \mathcal{T}_i : i \in I\rangle_T$ for the fusion system on T *generated* by the $\mathcal{T}_i$; that is the smallest fusion system on $\mathcal{T}$ containing each $\mathcal{T}_i$, which is just the intersection of all such fusion systems on T. Often we omit the "T" subscript.

Alperin's Fusion Theorem. *If $\mathcal{F}$ is a saturated fusion system then $\mathcal{F} = \langle Aut_{\mathcal{F}}(R) : R \in \mathcal{F}^{frc}\rangle_S$.*

Given a second fusion system $\mathcal{F}'$ on a p-group S', a *morphism* $\alpha : \mathcal{F} \to \mathcal{F}'$ of fusion systems is a group homomorphism $\alpha : S \to S'$, such that for each $P, Q \leq S$ and $\phi \in \hom_{\mathcal{F}}(P,Q)$, we have $\ker(\alpha_{|P})\phi \leq \ker(\alpha_{|Q})$, and the map $\phi^{\alpha} : x\alpha \mapsto x\phi\alpha$, $x \in P$, is in $\hom_{\mathcal{F}'}(P\alpha, Q\alpha)$. The kernel of the morphism α is its kernel $\ker(\alpha)$ as a homomorphism

$\alpha : S \to S'$ of groups. The morphism α is *surjective* if $\alpha : S \to S'$ is surjective, and for each $P', Q' \leq S'$ and $\phi' \in \hom_{\mathcal{F}'}(P', Q')$ there exists $P, Q \leq S$ and $\phi \in \hom_{\mathcal{F}}(P, Q)$ such that $P\alpha = P'$, $Q\alpha = Q'$, and $\phi^\alpha = \phi'$.

Define a subgroup T of S to be *strongly closed in S with respect to $\mathcal{F}$* if for each $P \leq T$ and $\phi \in \hom_{\mathcal{F}}(P, S)$, $P\phi \leq T$.

Example 5.3. Let G be a finite group, $S \in Syl_p(G)$, and $\mathcal{F} = \mathcal{F}_S(G)$. Let $H \trianglelefteq G$. Then $H \cap S \in Syl_p(H)$ and $H \cap S$ is strongly closed in S with respect to $\mathcal{F}$. Moreover if $\pi : G \to G/H$ is the natural map $\pi : g \mapsto gH$ then $\alpha = \pi_{|S} : S \to SH/H$ defines a surjective morphism of fusion systems $\alpha : \mathcal{F} \to \mathcal{F}_{SH/H}(G/H)$ with kernel $S \cap H$.

Lemma 5.4. *Let $\mathcal{F}'$ be a fusion system on S' and $\alpha : S \to S'$ a homomorphism of groups.*

(1) Assume $\ker(\alpha)$ *is strongly closed in S with respect to $\mathcal{F}$. Then for each $P, Q \leq S$ and $\phi \in \hom_{\mathcal{F}}(P, Q)$, the map $\phi^\alpha : P\alpha \to Q\alpha$ defined by $\phi^\alpha : x\alpha \to x\phi\alpha$ is an injective group homomorphism independent of the choice of representative in* $x \ker(\alpha_{|P})$.

(2) If $\alpha : \mathcal{F} \to \mathcal{F}'$ is a morphism of fusion systems then $\ker(\alpha)$ *is strongly closed in S with respect to $\mathcal{F}$.*

Proof. This is part of Exercise 2.

Notation 5.5. Given a category $\mathcal{C}$, an isomorphism $\alpha : A \to B$ in $\mathcal{C}$, and subobjects D, E of A, write $\alpha^* : \hom_{\mathcal{C}}(D, E) \to \hom_{\mathcal{C}}(D\alpha, E\alpha)$ for the map $\alpha^* : \phi \mapsto \alpha^{-1}\phi\alpha$.

For example if $\alpha : S \to S\alpha$ is an isomorphism of groups then $\mathcal{F}\alpha^*$ is the fusion system on $S\alpha^*$ with $\hom_{\mathcal{F}\alpha^*}(P\alpha, Q\alpha) = \hom_{\mathcal{F}}(P, Q)\alpha^*$ and $\alpha : \mathcal{F} \to \mathcal{F}\alpha^*$ is an isomorphism of fusion systems.

Example 5.6. Let $p = 2$ and S a quaternion group of order $m \geq 16$. Then S has two conjugacy classes Q_i^S, $i = 1, 2$, of quaternion subgroups of order 8. Let $\mathcal{U}$ be the universal fusion system on S. We define four subsystems of $\mathcal{U}$ on S.

First we write S for the system $\mathcal{F}_S(S)$. Next for $i = 1, 2$, set $\mathcal{F}_i = \langle Aut(Q_i) \rangle_S$. Finally set $\mathcal{F}_{1,2} = \langle Aut(Q_1), Aut(Q_2) \rangle_S$. Then S, $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_{1,2}$ are the four saturated fusion systems on S. This is assertion is part of Exercise 3, and follows from Alperin's Fusion Theorem and the fact that, up to conjugation in S, Q_1 and Q_2 are the only subgroups of S whose automorphism group is not a 2-group.

Moreover if α is an automorphism of S with $Q_1\alpha = Q_2$, then $\alpha : \mathcal{F}_1 \to \mathcal{F}_2$ is an isomorphism, so up to isomorphism there are exactly three saturated fusion systems on S.

Let q be an odd prime power with $(q^2-1)_2 = m$, where n_2 is the 2-share of an integer n. Then S is Sylow in the group $G = SL_2(q)$ and we will write $SL_2[m]$ for the fusion system $\mathcal{F}_S(G)$. By Exercise 3, $\mathcal{F}_{1,2} = SL_2[m]$. Similarly if $q = q_0$ is a square then G has a subgroup H with $O^2(H) \cong SL_2(q_0)$ and S Sylow in H, and if we choose notation so that $Q_1 \le H$ then $\mathcal{F}_S(H) = \mathcal{F}_1$.

In particular we see that the 2-fusion system of $SL_2(q)$ depends only on the 2-share of q^2-1, not on q.

Section 6. Local subsystems of fusion systems.

Let $\mathcal{F}$ be a fusion system on a p-group S. For $X \le S$, the *normalizer* in $\mathcal{F}$ of X is the fusion system $N_{\mathcal{F}}(X)$ on $N_S(X)$ such that for $P \le N_S(X)$, the $N_{\mathcal{F}}(X)$-maps from P into $N_S(P)$ are those $\phi \in \hom_{\mathcal{F}}(P, N_S(P))$ which extend to $\varphi \in \hom_{\mathcal{F}}(PX, N_S(P))$ such that φ acts on X. Moreover we say X is *normal* in $\mathcal{F}$ and write $X \trianglelefteq \mathcal{F}$ if $\mathcal{F} = N_{\mathcal{F}}(X)$. The *centralizer* $C_{\mathcal{F}}(X)$ of X in $\mathcal{F}$ is defined similarly.

We wish to develop a local theory of fusion systems analogous to the local theory of finite groups. The *local subsystems* of $\mathcal{F}$ in this theory are the systems $N_{\mathcal{F}}(X)$ for $1 \ne X \le S$. Of most interest are the normalizers of fully normalized subgroups, because of the following result of Puig:

Theorem 6.1. *(Puig) If $\mathcal{F}$ is saturated and $X \in \mathcal{F}^f$ then $N_{\mathcal{F}}(X)$ is saturated.*

Proof. See Theorem I.5.5 in [AKO].

It is not difficult to see that there is a largest subgroup of S normal in $\mathcal{F}$; we write $O_p(\mathcal{F})$ for this subgroup.

Lemma 6.3. *Let $\mathcal{F}$ be saturated and $Q \le S$. Then the following are equivalent:*

(1) $Q \trianglelefteq \mathcal{F}$.

(2) Q is strongly closed in S with respect to $\mathcal{F}$ and contained in each member of $\mathcal{F}^{frc}$.

(3) There is a series $1 = Q_0 \le \cdots \le Q_n = Q$ such that for each $1 \le i \le n$, Q_i is strongly closed in S with respect to $\mathcal{F}$ and $[Q, Q_i] \le Q_{i-1}$.

Proof. See I.4.5 and I.4.6 in [AKO].

Lemma 6.4. *Assume $\mathcal{F}$ is saturated and $Q \le S$ is strongly closed in S with respect to $\mathcal{F}$. Then each of the following imply that $Q \trianglelefteq \mathcal{F}$.*

(1) Q is abelian.

(2) There is $Q_1 \le Q$ such that $Q_1 \trianglelefteq \mathcal{F}$, Q/Q_1 is abelian, and $Aut_{\mathcal{F}}(Q_1)$ is a p-group.

Proof. Condition (1) is sufficient by the equivalence of parts (1) and (3) of 6.3. Exercise 4 shows that (2) is sufficient.

Our fusion system $\mathcal{F}$ is said to be *constrained* if there exists a centric subgroup of S normal in $\mathcal{F}$. From Definition 5.1 a normal subgroup Q of S is centric if $C_S(Q) \le Q$. A *model* for $\mathcal{F}$ is a finite group G such that $S \in Syl_p(G)$, $\mathcal{F} = \mathcal{F}_S(G)$, and $F^*(G) = O_p(G)$.

Model Theorem. *Let $\mathcal{F}$ be constrained and saturated. Then*

(1) there exists a model G for $\mathcal{F}$, and

(2) if H is a model for $\mathcal{F}$ then the identity map ι on S extends to an isomorphism $\check{\iota} : G \to H$, and $\check{\iota}$ is unique up to an automorphism c_z of G for some $z \in Z(S)$.

Proof. This is essentially Proposition C in [BCGLO]; see also I.4.9 in [AKO].

Lemma 6.5. *Assume $\mathcal{F}$ is saturated and let $R \in \mathcal{F}^{frc}$. Then $N_{\mathcal{F}}(R)$ is constrained, so there is a model $G(R)$ for $N_{\mathcal{F}}(R)$, and and $R = F^*(G(R))$.*

Proof. As $\mathcal{F}$ is constrained and R is fully normalized, $N_{\mathcal{F}}(R)$ is saturated by 6.1. As $R \trianglelefteq N_{\mathcal{F}}(R)$ and R is centric, $N_{\mathcal{F}}(R)$ is constrained, so it possesses a model G by the Model Theorem. Now $R \trianglelefteq G$ and R is centric, so $C_S(R) = Z(R)$ and hence $C_G(R) = Z(R) \times O_{p'}(C_G(R))$. As G is a model, $F^*(G) = O_p(G)$, so $O_{p'}(C_G(R)) = 1$. Thus $Z(R) = C_G(R)$. Then $Aut_{\mathcal{F}}(R) = Aut_G(R) \cong G/Z(R)$. Finally as R is radical, $O_p(Aut_{\mathcal{F}}(R)) = Inn(R)$, so $O_p(G/Z(R)) = R/Z(R)$, and hence $R = O_p(G)$, completing the proof.

Section 7. Factor systems.

Let $\mathcal{F}$ be a fusion system on a p-group S and S_0 a subgroup of S strongly closed in S with respect to $\mathcal{F}$. Let $\mathcal{N} = N_{\mathcal{F}}(S_0)$, $S^+ = S/S_0$, and for $x \in S$, set $x^+ = Sx$. Let $\theta : S \to S^+$ be the natural map $\theta : x \mapsto x^+$, and define the fusion system $\mathcal{F}/S_0$ on S/S_0 as in Exercise 2. For $P, Q \le S$ and $\phi \in \hom_{\mathcal{F}}(P, Q)$, recall from Exercise 2 that $\phi^+ : P^+ \to Q^+$ is defined by $x^+\phi^+ = (x\phi)^+$. Further from Exercise 2, $\hom_{\mathcal{F}/S_0}(P^+, Q^+) = \{\phi^+ : \phi \in \hom_{\mathcal{N}}(PS_0, QS_0)\}$. Indeed by Exercise 2:

Lemma 7.1. *$\mathcal{F}/S_0$ is a fusion system on S^+ and $\theta : \mathcal{N} \to \mathcal{F}/S_0$ is a surjective morphism of fusion systems with kernel S_0.*

The fusion system $\mathcal{F}/S_0$ is the *factor system* of $\mathcal{F}$ modulo S_0.

Theorem 7.2. *If $\mathcal{F}$ is saturated then $\theta = \theta_{\mathcal{F},S_0} : \mathcal{F} \to \mathcal{F}/S_0$ is a surjective morphism of fusion systems with kernel S_0.*

Proof. See II.5.12 in [AKO].

Lemma 7.3. *Assume $\mathcal{F}$ is saturated and $\rho : \mathcal{F} \to \tilde{\mathcal{F}}$ is a surjective morphism of fusion systems with kernel S_0. Let $\tilde{S}$ be Sylow in $\tilde{\mathcal{F}}$ and define $\pi : S^+ \to \tilde{S}$ by $x^+\pi = x\rho$. Then $\pi : \mathcal{F}/S_0 \to \tilde{\mathcal{F}}$ is an isomorphism of fusion systems with $\theta\pi = \rho$.*

Proof. As $S_0 = \ker(\rho)$, the map $\pi : S^+ \to \tilde{S}$ is a well defined isomorphism with $\theta\pi = \rho$. Next for $\phi^+ \in \hom_{\mathcal{F}/S_0}(P^+, Q^+)$ and $x^+ \in P^+$, $(x^+\pi)\phi^{+\pi} = x^+\phi^+\pi = (x\phi)^+\pi = x\phi\rho = (x\rho)\phi^\rho$, so $(\mathcal{F}/S_0)\pi^* = \mathcal{F}\rho = \tilde{\mathcal{F}}$, and hence $\pi : \mathcal{F}/S_0 \to \tilde{\mathcal{F}}$ is an isomorphism from 5.5.

Theorem 7.4. *Assume $\mathcal{F}$ is saturated. Then the map $S_0 \mapsto \mathcal{F}/S_0$ is a bijection between the set of subgroups of S_0 strongly closed in S with respect to $\mathcal{F}$, and the set of isomorphism classes of homomorphic images of $\mathcal{F}$.*

Proof. This is a consequence of 7.2 and 7.3.

Lemma 7.5. *If $\mathcal{F}$ is saturated then $\mathcal{F}/S_0$ is saturated.*

Proof. See II.5.4 in [AKO].

From basic group theory, the homomorphic images of a group are parameterized by its normal subgroups, while from Theorem 7.4, the homomorphic images of a saturated fusion system are parameterized by its strongly closed subgroups. In a later section we will define the notion of a "normal subsystem" of a saturated fusion system $\mathcal{F}$. The Sylow group of a normal subsystem will be strongly closed. However in general there are strongly closed subgroups which are Sylow in no normal subsystem, and, as in the case of groups, a strongly closed subgroup can be Sylow in many normal subsystems. Thus homomorphic images of a saturated fusion system are *not* parameterized by its normal subsystems.

Section 8. Direct and central products of fusion systems.

In this section p is a prime and for $i = 1, 2$, $\mathcal{F}_i$ is a fusion system on a p-group S_i. Set $S = S_1 \times S_2$ and let $\pi_i : S \to S_i$ be the ith-projection. For $P_i, Q_i \leq S_i$ and $\phi_i \in \hom_{\mathcal{F}_i}(P_i, Q_i)$, define $\phi_1 \times \phi_2 : P_1 \times P_2 \to Q_1 \times Q_2$ by

$$\phi_1 \times \phi_2 : (x_1, x_2) \mapsto (x_1\phi_1, x_2\phi_2).$$

Define $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ to be the category whose objects are the subgroups of S, and for $P, Q \leq S$, $\hom_{\mathcal{F}}(P, Q)$ consists of the maps $\phi : P \to Q$ such that $\phi = (\phi_1 \times \phi_2)_{|P}$ for some $\phi_i \in \hom_{\mathcal{F}_i}(P_i, Q_i)$.

We call $\mathcal{F}_1 \times \mathcal{F}_2$ the *direct product* of $\mathcal{F}_1$ and $\mathcal{F}_2$.

Lemma 8.1. *(1) $\mathcal{F}_1 \times \mathcal{F}_2$ is a fusion system on S.*

(2) If $\mathcal{F}_i$ is saturated for $i = 1, 2$ then so is $\mathcal{F}$.

Proof. See 2.1 in [A5] for (1). See 2.7 in [5] or I.6.6 in [AKO] for (2).

Let $D \leq Z(\mathcal{F}_1) \times Z(\mathcal{F}_2)$ with $D \cap Z(\mathcal{F}_i) = 1$ for $i = 1, 2$. Define the *central product* of $\mathcal{F}_1$ and $\mathcal{F}_2$ with respect to D to be

$$\mathcal{F}_1 \times_D \mathcal{F}_2 = (\mathcal{F}_1 \times \mathcal{F}_2)/D.$$

Lemma 8.2. *Set $S^+ = S/D$ and assume $\mathcal{E}$ is a fusion system on S^+ and for $i = 1, 2$, $\mathcal{E}_i$ is a saturated subsystem of $\mathcal{E}$ on S_i^+ such that the map $s_i \mapsto s_i^+$ is an isomorphism of $\mathcal{F}_i$ with $\mathcal{E}_i$. Assume*

(1) for $i = 1, 2$, for each $P \in \mathcal{E}_i^{fc}$, and for each $\phi \in Aut_{\mathcal{E}_i}(P)$, ϕ extends to $\hat{\phi} \in Aut_{\mathcal{E}}(PS_{3-i}^+)$ with $\hat{\phi} = 1$ on S_{3-i}^+; and

(2) $\mathcal{E} = \langle \hat{\phi} : \phi \in Aut_{\mathcal{E}_i}(P), P \in \mathcal{E}_i^{fc}, i = 1, 2\rangle$.

Then $\mathcal{E} = \mathcal{F}_1 \times_D \mathcal{F}_2$.

Proof. This is 2.9.6 in [A5].

Definition 8.3. Let $\mathcal{E}$ be a fusion system and $\mathcal{E}_i$ a subsystem of $\mathcal{E}$ on E_i for $i = 1, 2$. We say that $\mathcal{E}_1$ *centralizes* $\mathcal{E}_2$ if for $i = 1, 2$, $\mathcal{E}_i \leq C_{\mathcal{F}}(E_{3-i})$. In that event, $\mathcal{E}$ contains the subsystem $\hat{\mathcal{E}}_i = E_{3-i} * \mathcal{E}_i$ which is a central product of E_{3-i} and $\mathcal{E}_i$.

Lemma 8.4. *Let $\mathcal{E}$ be a fusion system on E such that $\mathcal{F}_i$ is a saturated subsystem of $\mathcal{E}$ for $i = 1, 2$ such that $\mathcal{F}_1$ centralizes $\mathcal{F}_2$. Then $\langle \hat{\mathcal{F}}_1, \hat{\mathcal{F}}_2 \rangle_{S_1 S_2}$ is a central product of $\mathcal{F}_1$ and $\mathcal{F}_2$.*

Proof. Let $T = S_1 S_2 \leq E$. As $\mathcal{F}_1$ centralizes $\mathcal{F}_2$, $[S_1, S_2] = 1$, so there is a surjection $\rho : S \to T$ such that $\pi_i \iota_i = \rho$ for $i = 1, 2$, where ι_i is the identity map on S_i. Set $D = \ker(\rho)$ and $S^+ = S/D$. Then $\xi : s^+ \to s\rho$ is an isomorphism of S^+ with T which restricts to an isomorphism of $\mathcal{F}_i^+$ with $\mathcal{F}_i$. As $\mathcal{F}_1$ centralizes $\mathcal{F}_2$, condition (1) of 8.2 is satisfied. Hence the lemma follows from 8.2.

Section 9. Normal subsystems of fusion systems.

In this section p is a prime, $\mathcal{F}$ is a saturated fusion system on a p-group S, and T is a subgroup of S strongly closed in S with respect to $\mathcal{F}$.

Definition 9.1. There are at least three equivalent definitions of an $\mathcal{F}$-invariant subsystem; here is one of them. A subsystem $\mathcal{E}$ of $\mathcal{F}$ on a strongly closed subgroup T of S is said to be *$\mathcal{F}$-invariant* if $Aut_{\mathcal{F}}(T) \leq Aut(\mathcal{E})$ and for each $P \leq T$ and each $\alpha \in \hom_{\mathcal{F}}(P, S)$, there exists $\varphi \in Aut_{\mathcal{F}}(T)$ and $\phi \in \hom_{\mathcal{E}}(P\varphi, T)$ such that $\alpha = \varphi\phi$.

Lemma 9.2. *Assume $\mathcal{E}$ is $\mathcal{F}$-invariant on T and $\mathcal{D}$ is a subsystem of $\mathcal{F}$ on D. Then*

(1) $\mathcal{E} \cap \mathcal{D}$ is a $\mathcal{D}$-invariant subsystem of $\mathcal{D}$ on $T \cap D$, and

(2) if $\mathcal{D}$ is $\mathcal{F}$-invariant then $\mathcal{E} \cap \mathcal{D}$ is $\mathcal{F}$-invariant on $T \cap D$.

Proof. This is 3.6 in [A4]; the proof is straight forward if one uses one of the other definitions of invariance. Here $\mathcal{E} \cap \mathcal{D}$ is the fusion system on $T \cap D$ such that for $P \leq T \cap D$, $\hom_{\mathcal{E}\cap\mathcal{D}}(P, T \cap D)$ consists of those $\phi : P \to T \cap D$ such that ϕ is both a $\mathcal{E}$-map and a $\mathcal{D}$-map.

Definition 9.3. There are at least three notions of "normal subsystem" in the literature. We will adopt the convention in [AKO] and [Cr]. Define a subsystem $\mathcal{E}$ of $\mathcal{F}$ on a strongly closed subgroup T to be *weakly normal* in $\mathcal{F}$ if $\mathcal{E}$ is $\mathcal{F}$-invariant and saturated. Define $\mathcal{E}$ to be *normal* in $\mathcal{F}$ if $\mathcal{E}$ is weakly normal and satisfies condition (N1):

(N1) For each $\phi \in Aut_{\mathcal{E}}(T)$, ϕ extends to $\hat{\phi} \in Aut_{\mathcal{F}}(TC_S(T))$ such that $[\hat{\phi}, C_S(T)] \leq Z(T)$.

We write $\mathcal{E} \trianglelefteq \mathcal{F}$ to indicate that $\mathcal{E}$ is normal in $\mathcal{F}$.

Example 9.4. Let G be a finite group, $S \in Syl_p(G)$, and $H \trianglelefteq G$. Then $\mathcal{F}_{S\cap H}(H) \trianglelefteq \mathcal{F}_S(G)$. See Proposition I.6.2 in [AKO] for a proof.

From 9.2.2 the intersection of invariant subsystems of $\mathcal{F}$ is $\mathcal{F}$-invariant. Unfortunately the intersection of normal subsystems is *not* in general normal. This not too serious a problem however, since it develops that the intersection of a pair of normal subsystems in not quite the right candidate for the greatest lower bound for the pair.

Theorem 9.5. *Let $\mathcal{E}_i$ be a normal subsystem of $\mathcal{F}$ on T_i for $i = 1, 2$. Then there exists a normal subsystem $\mathcal{E}_1 \wedge \mathcal{E}_2$ on $T_1 \cap T_2$ contained in $\mathcal{E}_1 \cap \mathcal{E}_2$. Moreover $\mathcal{E}_1 \wedge \mathcal{E}_2$ is the largest normal subsystem of $\mathcal{F}$ normal in $\mathcal{E}_1$ and $\mathcal{E}_2$.*

Proof. This is Theorem 1 in [A5].

Definition 9.6. Let Σ be a collection of subcategories of $\mathcal{F}$. Then by Theorem 9.5, the wedge of the set of all normal subsystems of $\mathcal{F}$ containing Σ is a normal subsystem of $\mathcal{F}$, which we denote by $[\Sigma]_{\mathcal{F}}$. We call this subsystem the *normal closure* of Σ in $\mathcal{F}$. Then we write $O^{p'}(\mathcal{F})$ for the normal closure $[S]_{\mathcal{F}}$ of S in $\mathcal{F}$. Thus $O^{p'}(\mathcal{F})$ is the smallest normal subsystem of $\mathcal{F}$ on S.

In a group the product of normal subgroups is again a normal subgroup. No theorem for fusion systems has been proved at that level of generality, but we do know the following:

Theorem 9.7. *Assume $\mathcal{E}_i$ is a normal subsystem of $\mathcal{F}$ on T_i for $i = 1, 2$. Assume further that $[T_1, T_2] = 1$. Then there exists a normal subsystem $\mathcal{E}_1\mathcal{E}_2$ of $\mathcal{F}$ on $T = T_1T_2$. Further if $T_1 \cap T_2 \leq Z(\mathcal{E}_i)$ for $i = 1, 2$, then $\mathcal{E}_1\mathcal{E}_2$ is a central product of $\mathcal{E}_1$ and $\mathcal{E}_2$.*

Proof. This is Theorem 3 in [A5].

Lemma 9.8. *Assume $\mathcal{E}_i$ is a normal subsystem of $\mathcal{F}$ on T_i for $i = 1, 2$, and $\mathcal{E}_1$ centralizes $\mathcal{E}_2$. Set $T = T_1T_2$, Then $\langle \hat{\mathcal{E}}_1, \hat{\mathcal{E}}_2 \rangle_T = \mathcal{E}_1\mathcal{E}_2$ is a central product of $\mathcal{E}_1$ and $\mathcal{E}_2$.*

Proof. As $\mathcal{E}_1$ centralizes $\mathcal{E}_2$, $[T_1, T_2] = 1$ and $T_1 \cap T_2 \leq Z(\mathcal{E}_i)$ for $i = 1, 2$. Thus $\mathcal{E}_1\mathcal{E}_2$ is a central product $\mathcal{C}$ by 9.7, while $\mathcal{C} = \langle \hat{\mathcal{E}}_1, \hat{\mathcal{E}}_2 \rangle_T$ by 8.4.

One of the weaknesses in our current theory of fusion systems is the lack of the notion of the "normalizer" or "centralizer" of an arbitrary subsystem of $\mathcal{F}$. We do have normalizers and centralizers of subgroups of S, and we also have the following result:

Theorem 9.9. *Let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$ on T. Then*

(1) the set of subgroups of S centralizing $\mathcal{E}$ has a largest member $C_S(\mathcal{E})$;

(2) $C_S(\mathcal{E})$ is strongly closed in S with respect to $\mathcal{F}$; and

(3) there exists a normal subsystem $C_{\mathcal{F}}(\mathcal{E})$ on $C_S(\mathcal{E})$ which centralizes $\mathcal{E}$.

Proof. See Theorem 4 in [A5].

Definition 9.10 The *hyperfocal subgroup* $\mathfrak{hyp}(\mathcal{F})$ of S is

$$\langle [x, \alpha] : x \in P \leq S,\ \alpha \in O^p(Aut_{\mathcal{F}}(P)) \rangle.$$

Theorem 9.11. *The hyperfocal subgroup $\mathfrak{hyp}(\mathcal{F})$ is strongly closed in S with respect to $\mathcal{F}$ and there exists a normal subgroup $O^p(\mathcal{F})$ of $\mathcal{F}$ on $\mathfrak{hyp}(\mathcal{F})$ such that $O^p(\mathcal{F}) = O^p(O^p(\mathcal{F}))$.*

Proof. See for example 7.7 in [A5].

Section 10. The generalized Fitting subsystem of a fusion system.

In this section p is a prime, $\mathcal{F}$ is a saturated fusion system on a p-group S.

We've finally reached the point where it becomes possible to define the notions of simple and quasisimple fusion systems, and the generalized Fitting subsystem of a saturated fusion system. Compare these notions to the corresponding notions for groups, appearing near the beginning of Section 1.

A saturated system $\mathcal{F}$ is *simple* if it has no nontrivial proper normal subsystems. Further $\mathcal{F}$ is *quasisimple* if $\mathcal{F} = O^p(\mathcal{F})$ and $\mathcal{F}/Z(\mathcal{F})$ is nontrivial and simple.

As in the case of groups, subnormality for saturated fusions systems is the transitive extension of normality for fusion systems. Then the *components* of $\mathcal{F}$ are its subnormal quasisimple subsystems. Write $E(\mathcal{F})$ for the normal subsystem $[\mathrm{Comp}(\mathcal{F})]_{\mathcal{F}}$ of $\mathcal{F}$ generated by the set $\mathrm{Comp}(\mathcal{F})$ of components of $\mathcal{F}$. It turns out the $E(\mathcal{F})$ centralizes $O_p(\mathcal{F})$ so, using Theorem 9.7, we can form the normal subsystem $F^*(\mathcal{F}) = O_p(\mathcal{F})E(\mathcal{F})$ of $\mathcal{F}$; $F^*(\mathcal{F})$ is the *generalized Fitting subsystem* of $\mathcal{F}$. Compare the following theorem to Lemma 1.1 for groups.

Theorem 10.1. *(1) $F^*(\mathcal{F})$ is a normal subsystem of $\mathcal{F}$.*

(2) $E(\mathcal{F})$ is a central product of the components of $\mathcal{F}$.

(3) $F^*(\mathcal{F})$ is a central product of $O_p(\mathcal{F})$ and $E(\mathcal{F})$.

(4) $C_{\mathcal{F}}(F^*(\mathcal{F})) = Z(F^*(\mathcal{F}))$.

Proof. This is Theorem 6 in [A5].

E-Balance Theorem. *For each fully normalized subgroup X of S, $E(N_{\mathcal{F}}(X)) \leq E(\mathcal{F})$.*

Proof. This is Theorem 7 in [A5].

Lemma 10.2. *$\mathcal{F}$ is constrained if and only if $E(\mathcal{F}) = 1$.*

Proof. Suppose $\mathcal{F}$ is constrained. Then there is a subgroup R of S normal in $\mathcal{F}$ such that $C_S(R) \leq R$. As $R \trianglelefteq \mathcal{F}$, we have $R \leq O_p(\mathcal{F})$, so R centralizes $E(\mathcal{F})$ by 10.1.3. Thus if $\mathcal{C}$ is a component of $\mathcal{F}$ on T then $T \leq C_S(R) \leq R \leq C_S(T)$, so T is abelian. But then $T \trianglelefteq \mathcal{C}$ by 6.4.1, so as $\mathcal{C}/Z(\mathcal{C})$ is simple, $T = Z(\mathcal{C})$. Then from Definition 9.10, $\mathfrak{hyp}(\mathcal{C}) = 1$, a contradiction as $\mathcal{C} = O^p(\mathcal{C})$ so $\mathfrak{hyp}(\mathcal{C})$ is Sylow in $\mathcal{C}$. Thus $\mathcal{F}$ has no components, so $E(\mathcal{F}) = 1$.

Conversely assume $E(\mathcal{F}) = 1$. Then $F^*(\mathcal{F}) = O_p(\mathcal{F}) = R$, and hence by 10.1.4, $C_S(R) \leq R$, so R is centric, and hence $\mathcal{F}$ is constrained.

Compare the following definitions to the corresponding definitions for groups in Definition 2.1.

Definition 10.3. Define $\mathcal{F}$ to be of *characteristic p-type* if for each $1 \neq X \in \mathcal{F}^f$, $N_{\mathcal{F}}(X)$ is constrained. Define $\mathcal{F}$ to be of *component type* if for some $X \in \mathcal{F}^f$ of order p, $E(C_{\mathcal{F}}(X)) \neq 1$.

Dichotomy Theorem for Fusion Systems. *Each saturated fusion system is either of characteristic p-type or of component type.*

Proof. Assume $\mathcal{F}$ is not of characteristic p-type. Then from 10.3, there exists $1 \neq U \in \mathcal{F}^f$ such that $\mathcal{N} = N_{\mathcal{F}}(U)$ is not constrained. By 6.1, $\mathcal{N}$ is saturated, so by 10.2, $E(\mathcal{N}) \neq 1$. Let T be Sylow in $\mathcal{N}$ and X a subgroup of U of order p normal in T. Then X is fully normalized in $\mathcal{N}$ and centralizes $E(\mathcal{N})$. Therefore (cf. 10.3 in [A5]), $E(\mathcal{N}) = E(N_{\mathcal{N}}(X))$. Let $\mathcal{C} = C_{\mathcal{F}}(X)$; replacing U and X by suitable conjugates and appealing to 5.2.2, we may assume $X \in \mathcal{F}^f$ and $U \in \mathcal{C}^f$. Moreover $E(C_{\mathcal{N}}(X)) = E(N_{\mathcal{C}}(U))$, so by E-Balance, $1 \neq E(N_{\mathcal{C}}(U)) \leq E(\mathcal{C})$. Hence $\mathcal{F}$ is of component type by 10.3.

Section 11. Tightly embedded subsystems.

With a basic theory of fusion systems in place, we are now in a position to attempt to translate the major pieces of the classification of the simple groups of component type into analogous steps in a program to classify the simple saturated 2-fusion systems of component type. Thus we need analogues of tightly embedded subgroups, the Component Theorem, the Classical Involution Theorem, and Walter's Theorem, in the category of saturated 2-fusion systems. Also, at some point, various standard form problems need to be solved. Most of this work is in preliminary form, so "proofs" and possibly even statements of some of the "theorems" may need to be modified.

We begin with a translation of the notion of a tightly embedded subgroup into a corresponding notion for fusion systems. Recall from Definition 3.1 that if G is a finite group, then a subgroup K of G is *tightly embedded* in G if K is of even order, but the intersection of any pair of distinct conjugates of K is of odd order. This formulation does not translate well into a statement about fusion systems, but there are equivalent formulations which do translate:

Lemma 11.1. *Let G be a finite group and K a subgroup of G of even order. Then the following are equivalent:*

(1) K is tightly embedded in G.

(2) For each nontrivial 2-subgroup X of K, $X^G \cap K = X^{N_G(K)}$ and $N_G(X) \leq N_G(K)$.

(3) For each involution x in K, $x^G \cap K = x^{N_G(K)}$ and $C_G(x) \leq N_G(K)$.

Proof. This is Exercise 5.

Now let $\mathcal{F}$ be a saturated fusion system on a p-group S.

Definition 11.2. A *tightly embedded subsystem* of $\mathcal{F}$ is a saturated subsystem $\mathcal{Q}$ of $\mathcal{F}$ on a nontrivial fully normalized subgroup Q of S, satisfying the following three conditions:

(T1) For each $1 \neq X \in \mathcal{Q}^f$ and $\alpha \in \mathfrak{A}(X)$, $O^{p'}(N_{\mathcal{Q}}(X))\alpha^* \trianglelefteq N_{\mathcal{F}}(X\alpha)$.

(T2) For each subgroup X of Q of order p, $X^{\mathcal{F}} \cap Q = X^{Aut_{\mathcal{F}}(X)\mathcal{Q}}$.

(T3) $Aut_{\mathcal{F}}(Q) \leq Aut(\mathcal{Q})$.

Here $X^{Aut_{\mathcal{F}}(Q)\mathcal{Q}} = \{X\varphi\phi : \varphi \in Aut_{\mathcal{F}}(Q) \text{ and } \phi \in \hom_{\mathcal{Q}}(X\varphi, Q)\}$.

Example 11.3. Let G be a finite group and $S \in Syl_p(G)$. Let K be a tightly embedded subgroup of G such that $Q = S \cap K \in Syl_p(K)$ and $N_S(Q) \in Syl_p(N_G(K))$. Then $\mathcal{F}_Q(K)$ is tightly embedded in $\mathcal{F}_S(G)$.

Example 11.4. The converse of 11.3 is not in general true. Let q be an odd prime power, $G = L_2(q^2)$, and $S \in Syl_2(G)$. Then S is dihedral and G contains a subgroup K isomorphic to $PGL_2(q)$ containing S. Set $L = O^2(K)$, $Q = S \cap L$, and $\mathcal{Q} = \mathcal{F}_S(L)$. Then $\mathcal{Q}$ is tightly embedded in $\mathcal{F} = \mathcal{F}_S(G)$, but (unless $q = 3$) L is not tightly embedded in G. Similarly S has a second dihedral subgroup Q' of index 2 in S and $\mathcal{Q}' = \mathcal{F}_{Q'}(Q')$ is tightly embedded in $\mathcal{F}_S(K)$ but Q' is not tightly embedded in K.

Definition 11.5. A subgroup Q of S is *$\mathcal{F}$-semistable* if for each $P \leq Q$ and $\phi \in \hom_{\mathcal{F}}(P, S)$, $P\phi$ is contained in some conjugate of Q.

In the remainder of the section we focus on 2-fusion systems, so from now on we assume $p = 2$.

Theorem 11.6. *Assume $\mathcal{Q} = O^{2'}(\mathcal{Q})$ is a saturated subsystem of $\mathcal{F}$ on a nontrivial fully normalized subgroup Q of S. Assume $\Phi(Q) \neq 1$. Then the following are equivalent:*

(1) $\mathcal{Q}$ is tightly embedded in $\mathcal{F}$ and Q is $\mathcal{F}$-semistable.

(2) $\mathcal{Q}$ is subnormal in $\mathcal{F}$ and $Q \cap R = 1$ for each $R \in Q^{\mathcal{F}} - \{Q\}$.

Proof. This is Theorem 1 in the preprint [A6].

At first glance it is not clear whether Theorem 11.6 is very interesting; after all, the condition in 11.6.1 that Q is $\mathcal{F}$-semistable appears fairly strong. However the next result says that if $m_2(Q) > 1$, then, with the exception of some cases where Q is "small", the semistability assumption is unnecessary.

Theorem 11.7. *Assume $\mathcal{Q} = O^{2'}(\mathcal{Q})$ is tightly embedded in $\mathcal{F}$ with Sylow group Q. Then one of the following holds:*

(1) $\mathcal{Q}$ is subnormal in $\mathcal{F}$.

(2) $\Phi(Q) = 1$.

(3) $m_2(Q) = 1$.

(4) Q is dihedral.

(5) $\mathcal{Q} = Q$ and Q has a unique abelian subgroup X of index 2 such that $\Phi(X) \neq 1$ and X is inverted by the members of $Q - X$. Moreover X is tightly embedded and subnormal in $\mathcal{F}$, and $Q \cap O_2(\mathcal{F}) = X$.

(6) $\mathcal{Q} = Q$ appears in one of three special cases.

Proof. This is Theorem 2 in the preprint [A6].

A number of other results on tightly embedded subsystems (corresponding to theorems on groups in [A1] and [A2]) are necessary to prove and exploit the Component Theorem, but our time is limited, so we will move on to the next topic.

Section 12. Toward a Component Theorem.

In this section, $\mathcal{F}$ is a saturated fusion system on a 2-group S, and $\mathcal{C}$ is a quasisimple subsystem of $\mathcal{F}$ on T. We will eventually give a definition of a "standard subsystem" of $\mathcal{F}$, analogous to the definition of a standard subgroup of a finite group appearing in Definition 3.1. We will also briefly discuss preliminary work aimed at proving a fusion theoretic version of the Component Theorem appearing in Section 3. We begin with some notation related to that in 1.4.

Notation 12.1. Set $Q_0 = C_S(T)$ and let $\mathcal{X} = \mathcal{X}(\mathcal{C})$ be the set of nontrivial subgroups X of Q_0 centralizing $\mathcal{C}$. Let $\tilde{\mathcal{X}} = \tilde{\mathcal{X}}(\mathcal{C})$ consist of those $X \in \mathcal{X}$ such that for some $\alpha \in \mathfrak{A}(X)$, $\mathcal{C}\alpha^*$ is a component of $N_{\mathcal{F}}(X\alpha)$. Let $\mathcal{I}$ be the set of involutions in $\tilde{\mathcal{X}}$. Let $\rho(\mathcal{C})$ consist of the pairs $(t\alpha, \mathcal{C}\alpha^*)$ such that $t \in \mathcal{I}(\mathcal{C})$ and $\alpha \in \mathfrak{A}(t)$. Let $\rho_0(\mathcal{C})$ consist of the $(t_1, \mathcal{C}_1)$ in $\rho(\mathcal{C})$ such that $C_{C_S(t_1)}(\mathcal{C}_1)^\# \subseteq \tilde{\mathcal{X}}(\mathcal{C}_1)$. Write $\mathfrak{C}$ for the set of quasisimple subsystems $\mathcal{C}$ of $\mathcal{F}$ such that $\mathcal{I}(\mathcal{C}) \neq \varnothing$. For $\mathcal{C} \in \mathfrak{C}$, write $[\mathcal{C}]$ for the set of $\mathcal{D} \in \mathfrak{C}$ such that $\mathcal{C}$ is a homomorphic image of $\mathcal{D}$, and set $[\mathfrak{C}] = \{[\mathcal{C}] : \mathcal{C} \in \mathfrak{C}\}$.

Compare the following result to Lemma 1.5 on groups.

Lemma 12.2. *Assume $E_4 \cong \langle t, a\rangle \in \mathcal{X}$ with $t \in \mathcal{I}$. Let $\alpha \in \mathfrak{A}(a)$, $\bar{a} = a\alpha$, $\bar{t} = t\alpha$, $\bar{\mathcal{C}} = \mathcal{C}\alpha^*$, and $\bar{\mathcal{F}} = C_{\mathcal{F}}(\bar{a})$. Then there exists a component $\mathcal{D}$ of $\bar{\mathcal{F}}$ such that one of the following holds:*

(1) $\mathcal{D} = \bar{\mathcal{C}}$.

(2) $\bar{t}$ acts on but does not centralize $\mathcal{D}$, and $\bar{\mathcal{C}}$ is a component of $C_{\mathcal{D}}(\bar{t})$.

(3) $\mathcal{D} \neq \mathcal{D}^t$, $\bar{\mathcal{C}} = E(C_{\mathcal{D}\mathcal{D}^{\bar{t}}}(\bar{t}))$, and $\mathcal{D} \in [\mathcal{C}]$.

Proof. This is essentially 10.11.3 in [A5].

Next we adopt notation analogous to that in Notation 1.6:

Notation 12.3. Define $\mathcal{C}$ to be *maximal* in $\mathfrak{C}$ if whenever the setup of 12.2 arises, then conclusion (2) of Lemma 12.2 never occurs. For $\mathcal{C}_0 \in \mathfrak{C}$, define $[\mathcal{C}_0]$ to be *maximal* if each member of $[\mathcal{C}_0]$ is maximal.

When $\mathcal{C}$ is maximal, Lemma 12.2 can be used to show that $\mathcal{C}$ is a component in the centralizer of many involutions. For example we can often achieve the following setup:

Definition 12.4. Define $\Delta(\mathcal{C})$ to consist of those conjugates $\mathcal{C}_1 = \mathcal{C}\phi^*$ of $\mathcal{C}$ such that $\mathcal{C}_1$ centralizes $\mathcal{C}$, $T^\# \subseteq \tilde{\mathcal{X}}(\mathcal{C}_1)$, and $(T\phi)^\# \subseteq \tilde{\mathcal{X}}(\mathcal{C})$. Then set $\mathcal{C}^\perp = \Delta(\mathcal{C}) \cup \{\mathcal{C}\}$.

Theorem 12.5. *Assume $\mathcal{C}$ is maximal, $T \in \mathcal{F}^f$, and $\mathcal{C}^\perp \neq \{\mathcal{C}\}$. Then either*
(1) $\mathcal{C}$ is a component of $\mathcal{F}$ and $\mathcal{C}^\perp$ is the set of $\mathcal{F}$-conjugates of $\mathcal{C}$, or
(2) $m_2(T) = 1$, $\mathcal{C}^\perp = \{\mathcal{C}, \mathcal{C}_1\}$ is of order 2, and $Z(\mathcal{C}) = Z(\mathcal{C}_1)$.

There is a “proof” of this result in my notes, but the proof is preliminary. Theorem 12.5 is a fusion system version of Theorem 5 in the paper [A1] where the Component Theorem for groups is proved; Theorem 5 is the most difficult result in [A1].

Here is one possible definition of the notion of a “standard subsystem”:

Definition 12.6. A *standard subsystem* of $\mathcal{F}$ is a quasisimple subsystem $\mathcal{C}$ of $\mathcal{F}$ on a fully normalized subgroup T of S, such that:
(S1) $\tilde{\mathcal{X}}$ contains a unique maximal member Q.
(S2) For each $1 \neq X \leq Q$ and $\alpha \in \mathfrak{A}(X)$, $\mathcal{C}\alpha^* \trianglelefteq N_{\mathcal{F}}(X\alpha)$.
(S3) If $1 \neq X \leq Q$ and $\beta \in \mathfrak{A}(X)$ with $X\beta \leq Q$ then $T\beta = T$.
(S4) $Aut_{\mathcal{F}}(T) \leq Aut(\mathcal{C})$.

Moreover define $\mathcal{C}$ to be a *nearly standard subsystem* of $\mathcal{F}$ with it satisfies conditions (S1)-(S3).

So far it is not clear that the definition of a “standard subsystem” in 12.6 is analogous to that of a standard subgroup in 3.1. But this is more apparent after the next result:

Proposition 12.7. *Assume $\mathcal{C}$ is a standard subsystem of $\mathcal{F}$, and choose notation so that Q is fully normalized. Then there exists a saturated subsystem $\mathcal{Q}$ of $\mathcal{F}$ on Q such that:*
(1) $\mathcal{Q} \trianglelefteq N_{\mathcal{F}}(T)$.
(2) $\mathcal{Q}$ centralizes $\mathcal{C}$.
(3) $\mathcal{Q}$ is tightly embedded in $\mathcal{F}$.
(4) For $1 \neq X \in \mathcal{F}^f$ with $X \leq Q$, $C_{N_{\mathcal{F}}(X)}(\mathcal{C}) = N_{\mathcal{Q}}(X)$.

Proof. There is a “proof” of this in my notes. Call $\mathcal{Q}$ the *centralizer* in $\mathcal{F}$ of $\mathcal{C}$.

Theorem 12.8. *Assume $[\mathcal{C}]$ is maximal in $[\mathfrak{C}]$. Then one of the following holds:*
(1) $\mathcal{C}$ is a component of $\mathcal{F}$.
(2) $\mathcal{C} \cong SL_2[m]$ for some $m \geq 16$, $\mathcal{C}^\perp = \{\mathcal{C}, \mathcal{C}_1\}$ is of order 2, and $Z(\mathcal{C}) = Z(\mathcal{C}_1) \leq Z(S)$.

(3) There exists $t \in \mathcal{I}(\mathcal{C})$ *and a component* $\mathcal{D}$ *of* $\mathcal{F}$ *contained in* $[\mathcal{C}]$ *such that* $\mathcal{D} \neq \mathcal{D}^t$ *and* $\mathcal{C} = E(C_{\mathcal{D}\mathcal{D}^t}(t))$.

(4) $\mathcal{C} \cong L_2[m]$ *for some* $m \geq 16$ *and there exists* $t \in \mathcal{I}(\mathcal{C})$, *an involution* a *centralizing* t *such that* $\langle a, t\rangle \in \mathcal{X}(\mathcal{C})$, *and a component* $\mathcal{D}$ *of* $\bar{\mathcal{F}}$ *such that* $SL_2[m] \cong \mathcal{D} \neq \mathcal{D}^{\bar{t}}$, $\mathcal{D}^{\perp} = \{\mathcal{D}, \mathcal{D}^{\bar{t}}\}$, $\bar{\mathcal{C}} = E(C_{\mathcal{D}\mathcal{D}^{\bar{t}}}(\bar{t}))$, *and* $Z(\mathcal{D}) = Z(\mathcal{D}^{\bar{t}}) \leq Z(S)$.

(5) $\rho(\mathcal{C}) = \rho_0(\mathcal{C})$ *and* $\mathcal{C}^{\perp} = \{\mathcal{C}\}$.

Proof. Again there is a "proof" of this in my notes which is preliminary. Theorem 12.8 should be compared to the Component Theorem for groups in section 3.

In 12.8.5 we would like to show that $\mathcal{C}$ is standard. If $Z(\mathcal{C}) \neq 1$ it can be shown that $\mathcal{C}$ is nearly standard. But what about condition (S4)?

Lemma 12.9. *Assume* $T \in \mathcal{F}^f$ *and set* $\Sigma = N_{Aut_{\mathcal{F}}(QT)}(T)$. *Then*

(1) $Aut_{\mathcal{F}}(T) = Aut_{\Sigma}(T)$.

(2) Σ *acts on* Q_0.

(3) If $\sigma \in \Sigma$ *and* $X \in \tilde{\mathcal{X}}$ *with* $X\sigma \in \tilde{\mathcal{X}}$ *then* $\sigma_{|T} \in Aut(\mathcal{C})$.

(4) If some characteristic subgroup of Q_0 *is in* $\tilde{\mathcal{X}}$ *then* $Aut_{\mathcal{F}}(T) \leq Aut(\mathcal{C})$.

(5) Suppose $Q \leq Q_0$ *with* $Q^{\#} \subseteq \tilde{\mathcal{X}}$ *and* $|Q| > |Q_0 : Q|$. *Then* $Aut_{\mathcal{F}}(T) \leq Aut(\mathcal{C})$.

Proof. Exercise 6.

For example $Q_1 = [Q_0, Q_0]$ centralizes $\mathcal{C}$, so, given (S1) and (S2), if Q_0 is nonabelian then $Q_1 \in \tilde{\mathcal{X}}$, so (S4) follows from 12.9.4 in this case. Or if $Z(\mathcal{C}) = Z(T)$ then we can apply 12.9.4 to $Z(T)$. Thus in certain situations one can show that, when $\mathcal{C}$ is nearly standard, condition (S4) holds and hence $\mathcal{C}$ is standard, but I have no proof of this in general. It would be nice to have such a proof.

One might also attempt to verify (S4) for various choices of "known" $\mathcal{C}$ on an ad hoc basis.

Example 12.10 Assume $\mathcal{C}$ has a dihedral Sylow group T. Then $|T| \geq 8$ and $\mathcal{C}$ is the fusion system on T such that $Aut_{\mathcal{C}}(E) = Aut(E)$ for each 4-subgroup E of T. From this description it is trivial that $Aut(T) \leq Aut(\mathcal{C})$, so that (S4) holds. The fusion system $\mathcal{C}$ is $L_2[m] \cong SL_2[m]/Z(SL_2[m])$, where $SL_2[m]$ is described in 5.6 and $|T| = m/2$. Put another way, $\mathcal{C}$ is the fusion system of $L_2(q)$, where $(q^2-1)_2 = m$.

Similarly if $\mathcal{C}$ is $SL_2[m]$ then $Aut(T) \leq Aut(\mathcal{C})$.

However there is a more serious problem than the verification of condition (S4): In general in case (5) of Theorem 12.8, $\mathcal{C}$ need not be nearly standard. For if $\mathcal{C}$ is nearly standard then Q_0 acts on the unique maximal member Q of $\tilde{\mathcal{X}}$. In particular for $t \in Q$ and $u \in Q_0$, $t^u \neq tz$ for any $t \in T - Z(\mathcal{C})$. The following examples show this need not be the case:

Example 12.11. Consider Example 2.4 where $G = \langle t \rangle X_1 X_2$ is the wreath product of a simple group X_1 by an involution t, so that $C_{X_1X_2}(t) = \{xx^t : x \in X_1\} = L \cong X_1$. Let $S \in Syl_2(G)$ with $t \in S$ and $\mathcal{F} = \mathcal{F}_S(G)$; then $\mathcal{C} = \mathcal{F}_{S\cap L}(L)$ appears in case (5) of Theorem 12.8. But for u an involution in $Z(X_1 \cap S)$, $t^u = tz$ with $z = uu^t \in Z(S \cap L)$. Indeed both t and t^u centralize $\mathcal{C}$, but of course z does not.

Similarly in Example 2.3, G is the split extension of a group $X = X(q)$ of Lie type over a field of order $q = 2^{2e}$ by a field automorphism t, and $L = C_X(t) \cong X(2^e)$. Let z be an involution in $Z(S \cap L)$; then the root group U of z is in the center of $S \cap X$ and there is $u \in U$ with $t^u = tz$.

In summary, if one adopts the definition in 12.6 of a "standard subsystem" then in case (5) of Theorem 12.8, $\mathcal{F}$ need not have a standard subsystem. I'll suggest one possible way to avoid this problem in a moment.

But, as in the discussion near the end of Section 4 and Definition 2.2, even when standard subsystems exist, it might be better to prove a component theorem for systems that are not of even characteristic (ie. $N_{\mathcal{F}}(X)$ is not constrained for some $1 \neq X \trianglelefteq S$) or not of Baumann characteristic 2 (ie. $N_{\mathcal{F}}(X)$ is not constrained for some nontrivial fully normalized X centralizing $J(S)$); in such a theorem one would seek to establish the existence of a standard subgroup $\mathcal{C}$ on T such that $T \trianglelefteq S$ or Baum(S) acts on T, respectively. In the former case, one would need to classify fusion systems of even characteristic, which might be too difficult a problem. In the latter case I have no good ideas for proving a Baumann component theorem. However see Problem 16.9 for a possible fix.

In short it is not clear exactly what an optimal statement of a component theorem for fusion systems should be, although it is perhaps possible to build on the preliminary results discussed above to obtain a suitable theory.

Section 13. Tight split extensions.

In order to exploit the existence of a standard subsystem, we need some results on certain kinds of extensions of quasisimple fusion systems. Thus in this section, $\mathcal{F}_0$ is a saturated fusion system on a 2-group S_0. If $\mathcal{E}$ is a fusion system on E then the *weak closure* of a subgroup P of E is $\langle P^{\mathcal{F}} \rangle$, and P is *weakly closed* in E with respect to $\mathcal{E}$ if $P^{\mathcal{E}} = \{P\}$.

Definition 13.1. A *split extension* of $\mathcal{F}_0$ is a pair $(\mathcal{F}, Q)$ where $\mathcal{F}$ is a saturated fusion system on a 2-group S, $\mathcal{F}_0 \trianglelefteq \mathcal{F}$, $O^2(\mathcal{F}) = O^2(\mathcal{F}_0)$, and Q is a complement to S_0 in S. The extension $(\mathcal{F}, Q)$ is said to be *tight* if Q is tightly embedded in $\mathcal{F}$.

Lemma 13.2. *Assume $(\mathcal{F}, Q)$ is a split extension of $\mathcal{F}_0$ and Q is weakly closed in S with respect to $\mathcal{F}$. Then $Q \trianglelefteq \mathcal{F}$, so $\mathcal{F} = Q \times \mathcal{F}_0$.*

Proof. This is Exercise 7.

Lemma 13.3. *Assume $(\mathcal{F}, Q)$ is a tight split extension of $\mathcal{F}_0$.*

(1) If Q is nonabelian then $Q \trianglelefteq \mathcal{F}$, so $\mathcal{F} = Q \times \mathcal{F}_0$.

(2) If Q is noncyclic abelian and $\Phi(Q) \neq 1$ then the weak closure of Q in S with respect to $\mathcal{F}$ is normal in $\mathcal{F}$.

Proof. These results are proved in my notes. In (1), it can be shown that Q is weakly closed, so that (1) follows from 13.2.

Definition 13.4. Assume $\mathcal{F}_0$ is quasisimple. A *critical split extension* of $\mathcal{F}_0$ is a tight split extension $(\mathcal{F}, Q)$ of $\mathcal{F}_0$ such that Q is a 4-group. Further $\mathcal{F}_0$ is said to be *split* if there exists no nontrivial critical split extension of $\mathcal{F}_0$; that is for each such extension $(\mathcal{F}, Q)$, $\mathcal{F}$ is a central product of $C_S(\mathcal{F}_0)$ with $\mathcal{F}_0$.

Conjecture 13.5. Every quasisimple 2-fusion system is split.

Actually I have little evidence for the truth of the conjecture, but I would be surprised if it were not true.

Notice that if $\mathcal{F}_0$ is quasisimple and $(\mathcal{F}, Q)$ is a tight split extension of $\mathcal{F}_0$ such that Q does not centralize $\mathcal{F}_0$, then by Lemma 13.3.1, Q is abelian. Let W be the weak closure of Q in S with respect to $\mathcal{F}$. If $W \trianglelefteq \mathcal{F}$, then $Q \leq O_2(\mathcal{F})$ and $O_2(\mathcal{F})$ centralizes $\mathcal{F}_0$ by 10.1.3, contrary to our assumption that Q does not centralize $\mathcal{F}$. Thus W is not normal in $\mathcal{F}$, so by 13.3.2 either Q is cyclic or Q is elementary abelian. Assume Q is not cyclic;

then Q is generated by its 4-subgroups U, and for each such U, $(U\mathcal{F}_0, U)$ is a critical split extension of $\mathcal{F}_0$, so if $\mathcal{F}_0$ is split then $U\mathcal{F}_0$ is a central product. But then $\mathcal{F}$ is also a central product. We will exploit this observation in the next section.

Section 14. Standard form problems.

In this section $\mathcal{F}$ is a saturated fusion system on a 2-group S and $\mathcal{C}$ is a standard subsystem of $\mathcal{F}$. Let $\mathcal{Q}$ be the centralizer in $\mathcal{F}$ of $\mathcal{C}$ and Q the Sylow group of $\mathcal{Q}$. Define $\mathcal{F}$ to be *almost simple* if $F^*(\mathcal{F})$ is a nonabelian simple system.

Lemma 14.1. *One of the following holds:*

(1) $\mathcal{C} \trianglelefteq \mathcal{F}$.

(2) $\mathcal{C}$ *is simple and* $F^*(\mathcal{F}) = \mathcal{C} \times \mathcal{C}\varphi^*$ *with* $\mathcal{C}\varphi^* = F^*(\mathcal{Q})$.

(3) $\mathcal{F}$ *is almost simple and* $\mathcal{C} \leq F^*(\mathcal{F})$.

(4) $Q = \langle u \rangle$ *is of order 2 and* $F^*(\mathcal{F}) = \mathcal{D} \times \mathcal{D}^u$ *for some simple component* $\mathcal{D}$ *of* $\mathcal{F}$ *such that* $\mathcal{C} \cong \mathcal{D}$ *is a full diagonal subsystem of* $F^*(\mathcal{F})$. *That is* $QF^*(\mathcal{F})$ *is the wreath product of* $\mathcal{C}$ *with a group of order 2.*

Proof. This appears in my notes. In cases (1) and (2), $\mathcal{C}$ is a component of $\mathcal{F}$, and in (4), u is neither in the center of S nor centralizes the Baumann subgroup of S. The most interesting case is case (3).

Definition 14.2. Let $\mathcal{C}$ be quasisimple. The *standard form problem* for $\mathcal{C}$ is to determine all almost simple saturated 2-fusion systems $\mathcal{F}$ in which $\mathcal{C}$ is a standard subsystem of $\mathcal{F}$. Or perhaps this problem should be modified so as to demand that $\mathcal{C}$ centralizes an involution in the center of S, or that the Baumann subgroup of S normalizes $\mathcal{C}$, or that $m_2(N_S(T)) = m_2(S)$.

Theorem 14.3. *Assume* $\mathcal{C}$ *is split. Then one of the following holds:*

(1) $\mathcal{C}$ *is a component of* $\mathcal{F}$.

(2) $m_2(Q) = 1$.

(3) $\Phi(Q) = 1$.

Proof. As usual a proof appears in my notes. It depends heavily on the theory of tightly embedded subsystems.

In case (2), Q is either quaternion or cyclic. Further if Q is quaternion then as $\mathcal{Q}$ is tightly embedded in $\mathcal{F}$, a Classical Involution Theorem for fusion systems would determine $\mathcal{F}$. Thus if $\mathcal{C}$ is split, then in solving the standard form problem for $\mathcal{C}$, we may assume that Q is cyclic or elementary abelian. In the case of groups, if Q is noncyclic it turns out that Q is a 4-group; presumably the same thing is true for fusion systems.

Section 15. Quaternion fusion packets and Walter's Theorem.

Recall from section 4 that the Classical Involution Theorem gives a means for recognizing groups of Lie type and odd characteristic, and deals with the case where G has a tightly embedded subgroup with quaternion Sylow 2-groups. The definition of a "classical involution" appears in Definition 4.1. But in [A3] there is a much more general setup (called Hypothesis Ω in [A3]), and in [A3] the groups appearing in that setup are determined. Those groups include groups with a classical involution, but there are also many more examples. The definition of the setup is fusion theoretic, and is very close to the following hypotheses for fusion systems:

Definition 15.1. A *quaternion fusion packet* is a pair $\tau = (\mathcal{F}, \Omega)$ where $\mathcal{F}$ is a saturated fusion system on a finite 2-group S and Ω is an $\mathcal{F}$-invariant set of subgroups of S such that:

(1) There exists an integer m such that for each $K \in \Omega$, K has a unique involution $z(K)$ and K is nonabelian of order m.

(2) For each pair of distinct $K, J \in \Omega$, $|K \cap J| \leq 2$.

(3) If $K, J \in \Omega$ and $v \in J - Z(J)$ then $v^{\mathcal{F}} \cap C_S(z(K)) \subseteq N_S(K)$.

(4) If $K, J \in \Omega$ with $z = z(K) = z(J)$, $v \in K$, and $\phi \in \hom_{C_{\mathcal{F}}(z)}(v, S)$ then either $v\phi \in J$ or $v\phi$ centralizes J.

Example 15.2. Recall the discussion of the class $Chev^*(p)$ of groups of Lie type over fields of odd characteristic p (other than $L_2(q)$ and the Ree groups) and their *fundamental subgroups*. Let $G \in Chev^*(p)$, $S \in Syl_2(G)$, and Ω the set of subgroups $L \cap S$, for L a fundamental subgroup of G such that $L \cap S \in Syl_2(L)$. Set $\tau(G) = (\mathcal{F}_S(G), \Omega)$ and call $\tau(G)$ the *Lie packet* of G. Then $\tau(G)$ is a quaternion fusion packet.

I presume that each quaternion fusion packet is the packet of one of the groups appearing in the various theorems in [A3]. In particular the generic examples are the Lie fusion packets of members of $Chev^*(p)$, but there are also many other classes of examples. For example the 2-fusion systems of the groups $Sp_6(2)$ and $\Omega_8^+(2)$ admit fusion packets.

I have extensive notes on the problem of determining all quaternion fusion packets, and believe I will be able to complete the problem.

As in the case of groups, a classification of quaternion fusion packets would supply a characterization of the 2-fusion systems of the members of $Chev^*(p)$, as p ranges over the odd primes. Then we can hope to use that characterization to prove a fusion theoretic version of Walter's Theorem (cf. Section 4). Here is a possible statement of Walter's Theorem for fusion systems:

Let $\mathcal{F}$ be a simple saturated fusion system on a 2-group S and $\mathcal{C}$ be a quasisimple subsystem of $\mathcal{F}$ such $\mathcal{C}$ is the 2-fusion system of a member of $Chev^*(p)$ for some odd prime p (with a small number of exceptions) and $\mathcal{I}(\mathcal{C})$ is nonempty. Then either $\mathcal{F}$ is the 2-fusion system of a member of $Chev^*(p)$ for some odd prime p, or $\mathcal{F}$ is a Benson-Solomon system.

The Benson-Solomon systems are the only know simple exotic saturated 2-fusion systems. A saturated fusion system $\mathcal{E}$ on a finite p-group is *exotic* if there exists no finite group G such that $\mathcal{E}$ is the p-fusion system of G.

As far as I know, no one has made an attempt to prove a version of Walter's Theorem for fusion systems. I have not thought seriously about the problem.

Recall that Walter's Theorem would make it possible to avoid the standard form problems for the 2-fusion systems of members of $Chev^*(p)$.

Section 16. Some open problems.

Here are various open problems in the program to determine the simple saturated 2-fusion systems of component type, and then to use that theorem to simplify the existing treatment of simple groups of component type.

Problem 16.1. Revisit some of the difficult results in [A5] such as Theorems 9.5, 9.7, and 9.9 from section 9, and find simpler and more intuitive proofs of those theorems. That may involve coming up with simpler, more natural (but equivalent) definitions of the objects $\mathcal{E}_1 \wedge \mathcal{E}_2$, $\mathcal{E}_1\mathcal{E}_2$, and $C_{\mathcal{F}}(\mathcal{E})$ appearing in those results. Ellen Henke has already made such an improvement in Theorem 5 of [A5], a result which I've not mentioned explicitly, but have used here implicitly.

Problem 16.2. Given a (suitable) saturated subsystem of a saturated fusion system $\mathcal{F}$, define the *normalizer* or *centralizer* of the subsystem, and prove this system is saturated. Example 12.11 shows that this could be tricky or even impossible.

Problem 16.3. Given a simple 2-fusion system $\mathcal{F}$ on S, determine those simple groups G such that $\mathcal{F}_S(G) = \mathcal{F}$. Some such result would be necessary to turn a classification of simple 2-fusion systems of component type into a classification of simple groups of component type.

Problem 16.4. Solve the standard form problem for some given quasisimple system $\mathcal{C}$ or family of such systems. Perhaps it would first be best to solve the *Baumann standard form problem* for $\mathcal{C}$; that is assume $\mathcal{C}$ is normalized by the Baumann subgroup of the Sylow group of the oversystem $\mathcal{F}$ in which $\mathcal{C}$ is standard. For example Justin Lynd has solved the Baumann standard form problem for $L_2[m]$ in the case where a Sylow group Q of the centralizer of $\mathcal{C}$ is cyclic; this leaves the case where Q is noncyclic and elementary abelian, where the 2-fusion systems of the alternating groups A_{10} and A_{11} make an appearance. I believe Matt Welz, a student of Richard Foote, has some results on this problem.

Problem 16.5. Solve the standard form problem for each Benson-Solomon system, or perhaps just the Baumann standard form problem.

Problem 16.6. Prove each nearly standard subsystem is standard; that is verify condition (S4) of Definition 12.6.

Problem 16.7. Prove Conjecture 13.5 on critical split extensions of quasisimple systems. Or perhaps prove the conjecture for the known simple systems other than those of groups of Lie type and odd characteristic. Or a less ambitious goal is to prove the conjecture for some particular class of quasisimple systems.

Problem 16.8. State and prove a version of Walter's Theorem for fusion systems.

Problem 16.9. State and prove a Baumann Component Theorem. Define a saturated 2-fusion system $\mathcal{F}$ over S to be a of *Baumann component type* if for some involution t in the center of $\mathrm{Baum}(S)$, $E(C_{\mathcal{F}}(t)) \neq 1$. Define $\mathcal{C}$ to be a *Baumann* standard subsystem of $\mathcal{F}$ if $\mathcal{C}$ is a standard subsystem such that $\mathcal{I}(\mathcal{C}) \cap Z(\mathrm{Baum}(S)) \neq \varnothing$. Then a Baumann Component Theorem would say something like, with known exceptions, if $\mathcal{F}$ is of Baumann component type and all members of $\mathfrak{C}$ are simple of known type and not of type $Chev^*(p)$, then $\mathcal{F}$ has a Baumann standard subsystem. I don't have any good ideas of how to prove such a result. However I do have a few vague ideas about how to prove that, if all members of $\mathfrak{C}$ are simple of known type and not in $Chev^*(p)$, and if $E(C_{\mathcal{F}}(t)) \neq 1$ for some involution t with $m_2(C_S(t)) = m_2(S)$, then, with known exceptions, there exists a standard subsystem $\mathcal{C}$ on T such that $m_2(N_S(T)) = m_2(S)$. Such a result might suffice, or perhaps could be used to prove a Baumann component theorem. Also it would probably avoid the problem that, in the general case, standard subsystems need not exists.

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