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Lecture 5 (2×45 min, 8 pages)

Global forms of K-theory, cont.

$$\begin{tikzcd} L(ku) \arrow[r] \arrow[d] & ku \arrow[r] & ku^c \arrow[r] \arrow[d, "\simeq_h"] & b(ku) \arrow[d] \\ L(KU) \arrow[rr] & & KU \arrow[r] & b(KU) \end{tikzcd}$$

(b is better than R because def'd on point set level)

$$\begin{tikzcd} {} & {} \arrow[l] & b(ku) \arrow[d] \\ & KU \arrow[r] & b(KU) \end{tikzcd}$$

} ultra-comm ring spectra, so pullback ku^c is also.

$ku(V) = \mathcal{C}(S^V, Sym(V_{\mathbb{C}}))$

gives $ku(\mathbb{R}) \cong U$ in particular, via eigenvalue, eigenspace decomposition.

$\langle\ \rangle : R(G) \longrightarrow \pi_0^G(ku)$ next:

Let W be a complex G-rep'n (unitary). Choose a real rep'n V and a complex G-embedding $W \hookrightarrow V_{\mathbb{C}}$. Let

$\langle W \rangle \in \pi_0^G(ku)$ be

$$\begin{tikzcd} S^W \arrow[r] & ku(V) \\[-20pt] v \arrow[r, mapsto] & {[v, W]} \end{tikzcd} \qquad W \subset V_{\mathbb{C}} \subset Sym(V_{\mathbb{C}})$$

Facts:

$\langle W \rangle$ only depends on the iso class of W

$\langle W \rangle + \langle W' \rangle = \langle W \oplus W' \rangle$

$\alpha^* \langle W \rangle = \langle \alpha^* W \rangle$

If $H \leq G$ and $[G:H] < \infty$ then $tr_H^G \langle W \rangle = \langle tr_H^G W \rangle = \langle \mathbb{C}G \otimes_{\mathbb{C}H} W \rangle$

$\langle W \rangle \cdot \langle W' \rangle = \langle W \otimes W' \rangle$ (Not equal on representatives since $W \otimes W'$ lands in the linear part, but tensoring representatives lands in quadratic part, so a homotopy is needed.)

$P^m \langle W \rangle = \langle W^{\otimes m} \rangle$ in $\pi_0^{\Sigma_m \wr G}(ku)$

$N_H^G \langle W \rangle = \langle W^{\otimes G/H} \rangle$ (tensor induction)

$\Rightarrow$ Well defined additive

$$\begin{tikzcd} R^+(G) \arrow[r] \arrow[d] & \pi_0^G(ku) \\ R(G) \arrow[ur, dashed, "\text{induced}"'] \end{tikzcd}$$

Prop: $R(G) \xrightarrow{\langle\rangle} \pi_0^G(ku)$ is split mono & is iso for G finite.

For $\dim G > 0$ the discrepancy has to do with infinite index transfers, as with $H\mathbb{Z}$ and $H\underline{\mathbb{Z}}$.

Segal: Let $H \overset{i}{\hookrightarrow} G$ be mono hom of cpt Lie gps. There is a smooth induction

$$i_! : R(H) \longrightarrow R(G).$$

First Hom, that's ∞ dim, then work on it to get a virtual rep'n.
∃ character formula!
Warning: $\langle\rangle : R(\,) \longrightarrow \underline{\pi}_0(ku)$ does not take smooth induction to the homotopy theoretic transfer. So not a hom of global functors.
(Bob Oliver has a homotopy theoretic smooth induction May be useful.)

Ex: $G = SU(2) \overset{i}{\hookleftarrow} T$ Then $i_!(1) = 2$ but in $\pi_0^{SU(2)}(ku)$

$tr_T^{SU(2)}(1) \neq 2$; difference detected by $\dim : \underline{\pi}_0(ku) = \underline{\pi}_0(H\mathbb{Z}$,

N.B.

$$\begin{tikzcd}
 & H\mathbb{Z} \arrow[rr] & & b(H\mathbb{Z}) \\
ku \arrow[ur] \arrow[rr] & & ku^c \arrow[ur] \arrow[d] & \\
 & & \underbrace{KU} &
\end{tikzcd}$$

problem goes away here: really get $R(G)$ in degree 0.

⤷

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Periodic global K-theory

Homotopy types go back to Segal (and Atiyah?) but the precise version is (almost) that of Michael Joachim. C^*-alg make the construction more natural.

Ingredients: $Cl(V)$ = Clifford algebra of V (complexified)

$$= \mathbb{C} \otimes (\text{Tensor}(V) / (v \otimes v - \langle v,v \rangle \cdot 1))$$

$$\dim_{\mathbb{C}} Cl(V) = 2^{\dim_{\mathbb{R}}(V)}$$

Cl is a $\mathbb{Z}/2 = \{\text{even}, \text{odd}\}$-graded C^*-algebra

$$Cl(V) \otimes Cl(W) \cong Cl(V \oplus W)$$

(Likely $(\alpha \otimes v)^* = -\bar{\alpha} \otimes v$)

Next need compact operators & the Hilbert space should vary nicely with G.

$\mathcal{H}_V$ = Hilbert space completion of $\text{Sym}(V_{\mathbb{C}})$
$\cong L^2(V)$ (Joachim's version)

$\mathbb{Z}/2$ graded by even and odd

$\text{Sym}(V_{\mathbb{C}}) \otimes \text{Sym}(W_{\mathbb{C}}) \cong \text{Sym}((V \oplus W)_{\mathbb{C}})$ induces $\mathcal{H}_V \hat{\otimes} \mathcal{H}_W \cong \mathcal{H}_{V \oplus W}$

Def: $s = C_0(\mathbb{R})$ = cont. complex valued functions on $\mathbb{R}$ vanishing at ∞
(again $\mathbb{Z}/2$ graded by even/odd)

Def: $KU(V) := \text{map}_{\mathbb{Z}/2, C^*}(s, \mathcal{H}_V \otimes Cl(V))$

(Eigenvalues can cluster at 0, otherwise discrete)
$\exists$ config space model (which looks unnatural?)

N.B. s is <u>not</u> commutative as a $\mathbb{Z}/2$-graded object.

N.B. Graded here means individual homogeneous parts NOT their direct sum.

What Stefan would like to do is to write down the Bott class and its inverse, & multiply them to get 1. Even non-equivariantly.

<u>Next</u>, product of KU and $ku^c \longrightarrow KU$.

<u>Multiplication</u>: $s \xrightarrow{\Delta} s \otimes s$ exists; meaning slightly obscure.

Δ: morphism of $\mathbb{Z}/2$ graded C^* algebras.

For $f \in C_0(\mathbb{R})_{even}$

$$\Delta(f)(x,y) = \begin{cases} f(\sqrt{x^2+y^2}) & f \text{ even} \\ \frac{x+y}{\sqrt{x^2+y^2}} f(\sqrt{x^2+y^2}) & f \text{ odd} \end{cases}$$

[Identify $s \otimes s \cong C_0(\mathbb{R}^2)$

$g \otimes h \longmapsto ((x,y) \longmapsto g(x)h(y))$

Not a ~~[illegible]~~ $\mathbb{Z}/2$-graded iso : $s \otimes s$ is $\mathbb{Z}/2$ graded, odd·odd has a sign on left, not on right.

Cocomm & Coassoc!

(In the literature it is usually presented only on generating functions.)

$\mu_{V,W} : KU(V) \wedge KU(W) \longrightarrow KU(V \oplus W)$ is

$$C^*(s, H_V \otimes Cl(V)) \otimes C^*(s, H_W \otimes Cl(W)) \xrightarrow{\otimes}$$

$$C^*(s \otimes s, H_V \otimes Cl(V) \otimes H_W \otimes Cl(W)) \xrightarrow{\Delta^*}$$

$$C^*(s, H_{V \oplus W} \otimes Cl(V \oplus W))$$

This gives an ultra-commutative ring spectrum KU.

⤵

<u>Thm</u> (Joachim) Let G be a compact Lie group, V a faithful G-representation s.t. $Sym(V_\mathbb{C})$ is a complete universe (*)
Then

$$map(S^V, KU(V)) \underset{G}{\simeq} BUP(\mathcal{U}_G)$$

(paraphrase of Joachim's statement)

<u>Cor</u>: $\pi_0^G(KU) \cong [S^V, KU(V)]^G$ for V large

$$\cong \pi_0^G BUP$$

$$\cong R(G)$$

to be defined now

Then, the map $R(G) \xrightarrow{\langle\rangle} \pi_0^G(ku) \xrightarrow{\pi_0^G(j)} \pi_0^G(KU)$ <u>must</u> be an iso.

Def of the morphism of ultra-commutative ring spectra $ku \to KU$:
(à la Peter Teichner)

$$ku(V) = \mathcal{C}(S^V, Sym(V_\mathbb{C})) \longrightarrow \begin{matrix} KU(V) \\ = C^*(s, \mathcal{H}_V \otimes Cl(V)) \end{matrix}$$

This requires functional calculus:

you have a function on the spectrum of an operator & you produce the operator which has the effect on Spectrum.

$$fc : S^V \longrightarrow C^*(s, Cl(V))$$

is

$$fc(v)(g) = \begin{cases} g(|v|) \cdot 1 & g \text{ even} \\ g(|v|) \cdot \frac{v}{|v|} & g \text{ odd} \end{cases}$$

[Again C^* alg folk use 1 even & 1 odd function to generate & then say what the map is on these two only.
Even: $\frac{1}{1+t^2}$
Odd: $\frac{t}{1+t^2}$

(*) Is this automatic from the fact that V is faithful?
Such are cofinal in any case.

Define $j(V): ku(V) \longrightarrow KU(V)$ by

$$[v_1,\dots,v_n,E_1,\dots,E_n] \longmapsto \Big\{ g \mapsto \sum_{i=1}^{n} \underset{\substack{\uparrow\\ \text{proj onto}\\ E_i\\ \in \mathcal{H}_V}}{P_{E_i}} \otimes \underbrace{fc(v_i)(g)}_{\in Cl(V)} \Big\}$$

functional calculus applied to mult by v_i

$O(V)$ - equivariance not too hard.

Well defined uses $pr_{E_i} + pr_{E_j} = pr_{E_i \oplus E_j}$ since they're orthogonal

Similarity of $\Delta(f)$ and fc makes the ~~a~~ map multiplicative.

<u>Unit</u>

$$S^V \longrightarrow ku(V)$$

$$v \longmapsto [v, \mathbb{C}] \longmapsto \{ g \mapsto P_{\mathbb{C}} \otimes fc(v)(g) \}$$

<u>Note</u>: we're landing in finite rank, not just compact, operators.

<u>Intuitive picture</u>: over S^1 we have Clifford submodules supported at a finite subset w/ symmetry: at t & $-t$ you get opposite subsets whose sum is a graded submodule. At 0 you have an actual ~~grade~~ Clifford module. At ∞, new rep'n's appear.

① Real and unitary version ought to be possible

② Completion map is natural?, but no help w/ proof

Dress - Amitsur complex? / Real

Forget & Right adjoint

Snaith splitting
$$\begin{tikzcd} & ku \arrow[dr] & \\ \Sigma^\infty_+ P^{\mathbb{C}} \arrow[ur] \arrow[rr] & & KU \end{tikzcd}$$

$P^{\mathbb{C}}(V) = P(\mathrm{Sym}\, V_{\mathbb{C}})$ (avoids $\mathbb{C}^\infty \oplus \mathbb{C}^\infty \cong \mathbb{C}^\infty$ problems,
is a c.o.m.s. recall)

$$P = P^{\mathbb{C}} = B^{\otimes}_{gl} U(1)$$

In literature ∃ false equivariant claims of Snaith splitting.

$$P(\mathrm{Sym}(V_{\mathbb{C}}))_+ \wedge S^V \longrightarrow ku(V)$$

$$L \wedge v \longmapsto [v, L]$$

rank 1 part ($\dim L = 1$)
∃ rank filtration of $k(V)$ being studied by…

Bott class

$\beta \in \pi_2^e(\Sigma^\infty_+ P) = \pi_2(\Sigma^\infty_+ \mathbb{C}P^\infty)$ classifies $\gamma_{\mathbb{C}} - 1$
↑ tautological

Snaith $\underbrace{[\Sigma^\infty_+ P][\beta^{-1}]} \simeq KU$

$$\mathrm{hocolim}\,(\Sigma^\infty P \to S^{-2} \wedge \Sigma^\infty P \to S^{-4} \wedge \Sigma^\infty P \to \cdots)$$

∃ universal property: approp ring spectrum inverting this elt.
Colim ~~may~~ will not create this in global setting.
If we have Norms & powers, we have to invert more than just one class, all its norms & powers as well.

Østvær & Spitzweck
$G = A$ abelian cpt Lie $\Sigma^\infty_+ P(\mathcal{U}_A)[1/\beta] \simeq KU_A$
Their proof would work if you invented Bott classes for all irreducibles.

What should work globally?

Conj:
$$(\Sigma^\infty_+ P)\left[\frac{1}{\beta_{U(1),\mathbb{C}}}\right] \longrightarrow KU$$
is an $\mathcal{Ab}$-global equivalence

$\beta_{U(1),\mathbb{C}}$ = Bott class of tautological $U(1)$ bundle

inverting in this category inverts all norms & powers, etc.

∃ such localization in this category

Expect on G f.p. it inverts all norms of all restrictions.

Another property: KU is right induced! (Lashof & May & ...?)

(*) $KU^*_G(X) \cong K^*_G(X \times E(G,\text{cyclic}))$ (characters determine reps)

says it is in image of R.

$$\begin{tikzcd} \mathcal{GH} \arrow[r] & \mathcal{GH}_{cyc} \arrow[l, bend right, "L"'] \arrow[l, bend left, "R"] \end{tikzcd}$$

$\mathcal{Ab}$ is closure of cyc under products

Conj would then give $KU \simeq R_{\mathcal{Ab}}\left(\Sigma^\infty_+ P\left[\frac{1}{\beta_{U(1),\mathbb{C}}}\right]\right)$
as Real ultra comm ring spectra.
Extremely highly structured version of Snaith's theorem.

(*) Does HKR theory say higher chromatic theories are right induced?

— The End —