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GHaHT
p.1

Lecture 4 (2×45 min, 10 pages)

Examples today.

X orthogonal spectrum, G cpt Lie, $\pi_0^G(X) = \operatorname{colim}_{V \in \mathcal{U}_G} [S^V, X(V)]^G$ indep of choice of $\mathcal{U}_G$ up to preferred iso

$X \xrightarrow{f} Y$ is a global equivalence $\Leftrightarrow$ $\pi_k^G(f)$ iso $\forall k \in \mathbb{Z}$ $\forall G$ (in family)

$\mathcal{GH} = \mathcal{S}p[\text{gl equiv}^{-1}]$

Burnside category A: obj = cpt Lie gps

mor $A(G,K) = \operatorname{Nat}(\pi_0^G, \pi_0^K)$

Global functor : $F: A \to Ab$ additive

$\mathcal{S}p^0 \ni X \longmapsto \underline{\pi}_0(X) = \{\pi_0^G(X)\}$, a global functor

For calculations, need a basis: $A(G,K) \overset{\text{Thm:}}{=} \mathbb{Z}\langle tr_L^K \circ \alpha^* \mid [L,\alpha],\ K \geq L \xrightarrow{\alpha} G,\ |W_K(L)| < \infty \rangle$

Ex $\mathcal{GF} \longrightarrow \operatorname{Funct}(Rep^{op}, Sets)$
forget $+$, tr
Δ left adjoint

Σ^∞_+ freely builds in the susp structure

For $Y \in$ OrthogSpace $\exists$ unit $Y \longrightarrow \Omega^\bullet(\Sigma^\infty_+ Y)$ induces a morph of Rep^{op} functors

$$\underline{\pi}_0 Y \longrightarrow \underline{\pi}_0(\Omega^\bullet \Sigma^\infty_+ Y) = \underline{\pi}_0(\Sigma^\infty_+ Y)$$

adjoint to a morphism of global functors

Thm: $\Delta(\underline{\pi}_0 Y) \xrightarrow[\cong]{} \underline{\pi}_0 \Sigma^\infty_+ Y$ Pf: tomDieck splitting. //

(Classically $\pi_0 \Sigma^\infty Y$ is $\mathbb{Z}[\text{set } \pi_0 Y]$.)

More concrete, special case $\rightsquigarrow$

Ex: $B_{gl}G = L(V,-)/G$ global classifying space

$\underline{\pi}_0^K(B_{gl}G) = Rep(K,G)$ induced by fund class $\in \underline{\pi}_0^G(B_{gl}G)$

i.e.

$$\underline{\pi}_0 B_{gl}G \cong Rep(-,G)$$

$$\Rightarrow \underline{\pi}_0(\Sigma^\infty_+ B_{gl}G) = A(G,-)$$

In other words, proj gen's are the $\Sigma^\infty_+ B_{gl}G$.

choose a set of iso cl. of cpt Lie G, e.g subgps of $O(n)$'s.

Fact: ~~Cor~~: $\{\Sigma^\infty_+ B_{gl}G\}$ is a set of compact generators for $\mathcal{GH}$

$\mathcal{GH}(\Sigma^\infty_+ B_{gl}G, X) \cong \underline{\pi}_0^G(X)$ has as a corollary.

More special case: $G=e$, $O=V$ is our faithfull rep'n. Then $B_{gl}e = *$

So $\Sigma^\infty_+ B_{gl}e = S$,

$$V \longmapsto L(0,V)/e_+ \wedge S^V = S^V$$

So

$\underline{\pi}_0 S = A(e,-)$ is the classical Burnside ring.

Next class of examples, <u>global Borel theories</u>

Let E be a non-equivariant cohom. theory. Then $G \longmapsto E^0 BG$ is a global functor. (Feshbach for tr). This is realized by the right adjoint

$$\begin{tikzcd} \mathcal{GH} \arrow[r, "\text{forget}"] & \mathcal{SH} \arrow[l, bend right, "L"'] \arrow[l, bend left, "R"] \end{tikzcd}$$

No closed formula for $\underline{\pi}_0 LX$, but <u>Thm</u>: $\underline{\pi}_0^G RE \cong E^0 BG$ natural in G, iso

$\underline{\pi}_k^G RE \cong E^{-k} BG$ of global functors.

(requires a specific point set level construction of R. No $\rightarrow$

$$\begin{array}{ccc} \mathcal{GH}(\Sigma^\infty_+ B_{gl} G, RE) & = & \mathcal{SH}(U(\Sigma^\infty_+ B_{gl} G), E) \\ \| & & \| \\ \pi_0^G RE & & E^0 BG \end{array} \quad \text{formally.} \quad //$$

N.B. L is strong symm monoidal
R is lax — " — (at point set level)

(usual $\wedge$ for orthog spectra is derivable w.r.t. global equivalences)

Point set level model for R

E an orthog. spectrum. Let bE be an orthog spectrum by
$$(bE)(V) = map(L(V, \mathbb{R}^\infty), E(V)) \quad \text{w/ diag } O(V)\text{-action}$$

Then
$$(bE)(V) \wedge (bF)(W) \longrightarrow b(E \wedge F)(V \oplus W)$$

is

$$\begin{tikzcd}
map(L(V,\mathbb{R}^\infty), E(V)) \wedge map(L(W,\mathbb{R}^\infty), F(W)) \arrow[d] \arrow[r] & map(L(V \oplus W), (E \wedge F)(V \oplus W)) \\
map(L(V,\mathbb{R}^\infty) \times L(W,\mathbb{R}^\infty), E(V) \wedge F(W)) \arrow[ur, "{map(res_{V,W},\, i_{V,W})}"'] &
\end{tikzcd}$$

giving $bE \wedge bF \longrightarrow b(E \wedge F)$.

__Thm:__ If E is a non-equiv. Ω spectrum then bE is a global Ω-spectrum and b realizes R.

Adjunction unit: $E(V) \xrightarrow{const} map(L(V, \mathbb{R}^\infty), E(V))$

__Note__ if V is faithful for G then $L(V, \mathbb{R}^\infty)$ is an EG.

If R is a comm orthog ring spectrum then

$\underline{\pi}_0 R$ also has products, power operations, & norm maps.

(since $\Omega^\bullet R$ is a c.o.m.s.)

$$\begin{array}{c} map(S^V, R(V)) \wedge map(S^W, R(W)) \longrightarrow map(S^V \wedge S^W, R(V) \wedge R(W)) \\ \| \\ (\Omega^\bullet R)(V) \wedge (\Omega^\bullet R)(W) \xrightarrow{\mu_{V,W}} map(S^{V\oplus W}, R(V\oplus W)) \end{array}$$

Norms & power operations come from comm orth monoid space structure on $\Omega^\bullet R$.

$$\underset{G}{P^m([S^V \xrightarrow{f} R(V)])} = [\ \cancel{f^{\wedge m}}\ \underset{\Sigma_m \wr G}{(S^V)^m \xrightarrow{f^m} R(V)^m} \xrightarrow[\mu_{V V \ldots V}]{} R(V^m)\ ]$$

R comm orth ring spectrum $\Rightarrow$ the power operations in $\underline{\pi}_0\, bR$ are the power operations in $G \longmapsto R^0 BG$.

Global power functor more convenient than Tambara functors because the power operations have better explicit formulas. [They are equivalent categories.]

— BREAK —

Example: Constant global functors. M an abel gp.

(like constant Mackey functors: const in restr direction forces tr direction)

$\underline{M}(G) = M$, $\alpha^* = \mathrm{Id}$, tr_H^G = mult. by $\chi(G/H)$

Only check that isn't obvious is double coset. It is true.

$\exists$ Eilenberg-MacLane spectrum $H\underline{M} \in \mathcal{GH}$ w/ $\pi_k H\underline{M} \cong \begin{cases} \underline{M} & k=0 \\ 0 & k\neq 0 \end{cases}$

Most obvious point-set model is not the constant $H\underline{M}$! Is this the right thing? We define

$$H\mathbb{Z}(V) = \tilde{\mathbb{Z}}[S^V] = \text{reduced } \cancel{\text{function}} \text{ free abel gp on } S^V$$

$\uparrow$ (M-space easiest) (basept $= 0 \iff$ reduced)

non-eq E.M. space of type $(\mathbb{Z}, \dim(V))$

$$\tilde{\mathbb{Z}}[S^V] \wedge \tilde{\mathbb{Z}}[S^W] \longrightarrow \tilde{\mathbb{Z}}[S^{V\oplus W}]$$

$$\sum a_i v_i \wedge \sum b_j w_j \longmapsto \sum a_i b_j (v_i \wedge w_j)$$

Segal, Shimakawa: G finite $\Rightarrow \tilde{\mathbb{Z}}[S^V]$ is an equiv. E.M. space for the constant Mackey functor $\underline{\mathbb{Z}}$:

$\tilde{\mathbb{Z}}[V]$?

$$map^G(S^V, \tilde{\mathbb{Z}}[S^V]) \cong \mathbb{Z}$$

Proof fails for non-disrete G, so

$H\mathbb{Z}$ is a $\mathcal{F}in$-global Ω-spectrum

$\pi_0^G(H\mathbb{Z}) = \mathbb{Z}$ for G finite but $\underline{\pi}_0 H\mathbb{Z} \not\cong \underline{\mathbb{Z}}$.

Pf: Consider $G = SU(2) \geq T = \{\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}\}$ max torus

$$tr_T^{SU(2)}(1), \quad 2 \in \pi_0^{SU(2)}(H\mathbb{Z})$$

$SU(2)/T \cong \mathbb{C}P^1$ w/ $\chi(SU(2)/T) = 2$, so in $\underline{\mathbb{Z}}$ $tr(1) = 2$.

But here this fails: pass to geom f.p.

$$\Phi : \pi_0^{SU(2)}(H\mathbb{Z}) \longrightarrow \pi_0(\Phi^{SU(2)} H\mathbb{Z})$$

use original geom description of geom f.p., not $\tilde{E}\mathcal{P}$ version.

$$= \operatorname*{colim}_{V \in \mathcal{U}_{SU(2)}} [S^{V^{SU(2)}}, H\mathbb{Z}(S^V)^{SU(2)}]$$

Φ kills all transfers from proper subgroups, so $\Phi(tr_T^{SU(2)}(1)) = 0$.

$H\mathbb{Z}(S^V)$ points are finite sums of points, so to be $SU(2)$ invariant, we must have (since $SU(2)$ conn.)

$$H\mathbb{Z}(S^V)^{SU(2)} = H\mathbb{Z}(S^{V^{SU(2)}})$$

So

$$= \operatorname{colim}_{V \subseteq \mathcal{U}_{SU(2)}} \underbrace{[S^{V^{SU(2)}}, H\mathbb{Z}(S^{V^{SU(2)}})]}_{= \mathbb{Z}}$$

and so $\Phi(2) = 2 \neq 0$.

What does this mean? Comm product on $H\underline{\mathbb{Z}}$ less explicit than on $H\mathbb{Z}$. Which is the good $H\mathbb{Z}$?

[In fact, here $tr_T^{SU(2)}(1)$ and 1 are lin indep. Result is rank 2.
general result. = Burnside ring / finite index transfers = their index]

Closed form versions (e.g. $H\mathbb{Z}$) should not be ignored.
So perhaps $H\underline{\mathbb{Z}}$ is "wrong". Similar problems
$H\mathbb{Z}$ is the global delooping of $(\mathbb{Z}, +)$ so have built in infinite index norms freely, ignoring the existing Euler characteristic $\chi(G/H)$ norms.
Note $\exists$

$$H\mathbb{Z} \longrightarrow H\underline{\mathbb{Z}}$$

Global forms of top K-theory

7 different global homotopy types

$L(ku)$ homotopy ring spectrum but not point set level ring spectrum (ultra-comm ring spectrum)
$L(KU)$ — " —

$R(ku)$ ~~Greenlees global kar?~~ } both ring spectra
$\downarrow$
$R(KU)$ Borel K-theory

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$$\begin{tikzcd}
L(ku) \arrow[r] \arrow[d] & \underset{(\text{Segal})}{ku} \arrow[r] & \underset{\text{Greenlees}}{ku^c} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner\, hpb"] & R(ku) \arrow[d] \\
L(KU) \arrow[rr] & & \underset{\substack{\text{Segal, Joachim} \\ (*)}}{KU} \arrow[r] & R(KU)
\end{tikzcd} \qquad (3)$$

(1) ku : conn global
(2) ku^c : global conn

(*) Bohmann notes that Joachim's construction is global

(1) not cx oriented, brutally truncate homotopy below 0, can't have non-trivial Euler classes since they lie in ku^{pos} = homotopy$_{<0}$

(2) a global (equivariant) form of connective k-theory

(1) is the global group completion of comm orthog monoid space (as in $H\mathbb{Z}$)

New thing: delooping for non-finite G. what should we expect it to produce?

global deloopings are

(3) Right hand square is Greenlees' definition ?

$$\left[\begin{tikzcd} R(ku) \arrow[d] & b(ku) \arrow[d] \\ R(KU) \arrow[ur, phantom, "\text{is really}"] & b(KU) \end{tikzcd}\right.$$

Config space models (today), C^* alg models (tomorrow)

<u>Complex</u> case only. Real is possible.

Let $\mathcal{U}$ = cx Hermitian i.p. space of possibly infinite dim ($\cong \mathbb{C}^n$, $n \le \infty$)
For based space X let
$\mathcal{C}(X, \mathcal{U})$ = space of finite configurations on X labelled by pairwise orthogonal f.d. subspaces of $\mathcal{U}$

↷

$$= \coprod_{n \geq 0} X^n \times \mathrm{sub}^n(\mathcal{U}) / \sim$$

n-tuples of pairwise orthog subspaces

$$ku(V) = \mathcal{C}(S^V, \mathrm{Sym}(V_{\mathbb{C}}))$$ w/ diagonal $O(V)$-action

new ingredient, V in both; One gp at a time used fixed V in $\mathrm{Sym}(V_{\mathbb{C}})$ which was a G-universe. If we try to do that globally we get $\mathbb{C}^\infty \times \mathbb{C}^\infty \neq \mathbb{C}^\infty$. This is an easier fix than using operads.

$\mu_{V,W} : ku(V) \times ku(W) \longrightarrow ku(V \oplus W)$ is

$$[v_1, \dots, v_n, E_1, \dots, E_n] \times [w_1, \dots, w_m, F_1, \dots, F_m] \mapsto [v_i \wedge w_j, E_i \otimes F_j]$$

using ~~canonical~~ natural $\mathrm{Sym}(V_{\mathbb{C}}) \otimes \mathrm{Sym}(W_{\mathbb{C}}) \cong \mathrm{Sym}((V \oplus W)_{\mathbb{C}})$

$\dim : ku \longrightarrow H\mathbb{Z}$

$[v_1, \dots, v_n, E_1, \dots, E_n] \longmapsto \sum \dim(E_i) v_i$

morphism of ultra-comm ring spectrum.

$ku^c \to$ borel theory of $H\mathbb{Z}$, not to $H\mathbb{Z}$.

Thm: For finite G, $ku\langle G \rangle$ is G-equivariant connective K-theory.
Pf: Shimakawa's ∞ loop space machine. //

Tomorrow: $\mathrm{Rep}(G) \longrightarrow \pi_0^G(ku\langle G \rangle)$

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Question session

What does this have to do with K-theory?

$$ku(\mathbb{R}) = \mathcal{C}(u(1), \mathbb{C}^\infty) \cong U$$

$[\lambda_1, \dots, \lambda_n, E_1, \dots, E_n] \longmapsto \phi$ w/ eigenvalue, eigenspace pairs λ_i, E_i (and 1 outside)

similarly $\longleftarrow \phi$

(from a paper of Suslin, short paper, on C_2-equivariant alg. K-theory. probably known to Segal, etc.)

$\exists$ C^* alg version

Segal uses a constant universe: okay non-equivariantly & if you don't care about a product.
These two issues seem to arise together.

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tom Dieck splitting: X a based G-CW-spectrum

$(|G|<\infty)$ $\pi_*^G(\Sigma^\infty X) \cong \bigoplus_{(H)} \pi_*^{WH}(EWH_+ \wedge X^H)$ ← tom Dieck's version; $\Sigma^{\mathrm{ad}\,H}$ if $\dim G > 0$

$WH = W_G H$, Weyl group

cpt Lie G version has a twist which we should look up

$\cong$ (Adams iso)

$$\bigoplus_{(H)} \pi_*\left(EWH_+ \wedge \Sigma^{\mathrm{ad}\,H} \wedge X^H\right)/WH$$

diagonal WH action

$X = S^0 \rightsquigarrow$ Equivariant π_0 is Burnside ring

$$\pi_0^G S = \bigoplus_{(H)} \pi_0 B(WH)_+$$

dim $H > 0 \Rightarrow$ this info vanishes from π_0

$$\cong \bigoplus_{(H)} \mathbb{Z}$$

The map uses Wirthmüller, transfer, ... Not elementary.
(Segal ICM talk) (may have simple description)
(Recommend tom Dieck's original paper for excellent exposition.)

$H = G$ summand $=$ Geom f.p.
others are transfers