

Start again w/ the stable situation.

Previous approaches:

Lewis-May-~~St~~ 3 section encoding compatibility; change of group & universe functors

Greenlees & May: Completions of MU modules set up formalism to study norm maps, indexing on (G, V)

Böhmann: made more explicit, an equivalent category

Def: (formerly \mathcal{L}_* prefunctors)

An orthogonal spectrum consists of

- based spaces X_n , $n \geq 0$
- $O(n)$ actions on each X_n ,
- structure maps $\sigma_n: X_n \wedge S^1 \rightarrow X_{n+1}$

such that $\forall n, m \geq 0$

$$X_n \wedge S^m \xrightarrow{\sigma_n \wedge 1} X_{n+1} \wedge S^{m-1} \rightarrow X_{n+2} \wedge S^{m-2} \rightarrow \dots \rightarrow X_{n+m}$$

is $O(n) \times O(m)$ -equivariant.

Coordinate-free version requires Thom complexes over complements.

V an inner product space, $\dim V = n$, X orthog spectrum,

Def: $X(V) := L(\mathbb{R}^n, V)_{O(n)}^+ \wedge_{O(n)} X_n$ as an $C(V)$ space.

Of course X_n in disguise but now coordinate free.

Generalized structure maps

$$X(V) \wedge S^W \longrightarrow X(V \oplus W), \quad O(V) \times O(W) \text{ equivariant.}$$

No G mentioned yet, but implicit via reps:

If G cpt Lie and V is a G -rep then $X(V)$ has G action via $G \rightarrow O(V)$.

Note $X(\mathbb{R}^n) = X_n$ is fair statement since isomorphism is so canonical.

We get equivariant stable homotopy groups

Def: For $k \in \mathbb{Z}$

$$\pi_k^G(X) = \operatorname{colim}_{V \in \mathcal{U}_G} [S^{V+k}, X(V)] \quad (k \geq 0; [S^V, X(V+k)] \text{ if } k < 0)$$

\mathcal{U}_G ← complete
 ↑
 poset of f.d. G
 sub reps

Formalism allowing pairs $(G, \text{universe})$ possible, but applications unclear.

These are morphisms in approp category: $[S^k, X]$.

Def: $X \xrightarrow{f} Y$ morphism of orthog. spectra is a global equivalence if

$$\pi_k^G(f) : \pi_k^G X \longrightarrow \pi_k^G Y$$

is an isomorphism $\forall k \in \mathbb{Z}, \forall G$.

Can restrict G to various subsets (e.g. finite) to get other notions of equivalence.

Def: The global stable homotopy category is $\mathcal{SH} = \operatorname{Sp} [\text{gl. equiv.}]$.

Thm: The global equivalences are part of a proper, cofibrantly generated, topological stable monoidal model category. The fibrant objects are the global omega spectra (Ω -spectra), i.e. the X s.t. ~~$\forall G$~~ $\forall G$ and V, W G -reps, V faithful, the map

$$\tilde{\sigma}_{V,W} : X(V) \longrightarrow \operatorname{map}(S^W, X(V \oplus W))$$

is a G wk equivalence. (*)

(*)
key: Don't require a good homotopy type on non-faithful reps. Meaningful values are only attained on faithful reps. Requiring $\tilde{\sigma}_{V,W}$ wk equiv ...

$X \xrightarrow{f} Y$ gl. eq. $\Rightarrow f \langle G \rangle : X \langle G \rangle \rightarrow Y \langle G \rangle$ is a G -stable equiv.
 giving

$$u : \mathcal{Y}H \begin{matrix} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{matrix} Ho(G\text{-Sp})$$

not a recollement. Has both adjoints but they're not fully faithful.

Not every G homotopy type is part of a global family. The global spectra are always split, for example.

For G , take all subgps & quotients & work relative to that family. Still not a part of a recollement, L&R not fully faithful.

Proof in 2 steps: first a level model structure. Then localize G it is no longer just about $O(n)$ but about G sitting in, in many ways.

Have to produce generators: take all closed subgps of $O(n)$, and that means all cpt Lie G . So just accepting coordinate-free approach, use all G , works best.

Global family: closed under iso, subgp (closed), & quotients.

All cpt Lie
 All Finite. \mathcal{F}
 All Abelian^{cpt} Lie Gps
 All trivial Lie Gps

} most used examples.

$$\mathcal{Y}H_+^{\mathcal{F}} \begin{matrix} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{matrix} \mathcal{Y}H_{\mathcal{F}} \begin{matrix} \xleftarrow{L\mathcal{F}} \\ \longrightarrow \\ \xleftarrow{R\mathcal{F}} \end{matrix} Sp$$

Odd note: Can mix families to get perverse examples.

Examples to come, but we need language for their properties.

One now: \exists usual adjoint functor pair

$$\text{Orthog. Spaces} \begin{matrix} \xrightarrow{\Sigma_+^\infty} \\ \xleftarrow{\Omega^\circ} \end{matrix} \mathcal{S}_p$$

\downarrow
 Y

\downarrow
 X

$$(\Sigma_+^\infty Y)(V) := Y(V)_+ \wedge S^V$$

w/ diagonal $O(V)$ -action

$\&$ structure map

$$Y(V) \rightarrow Y(V \oplus W)$$

$$S^V \wedge S^W \xrightarrow{\cong} S^{V \oplus W}$$

$$(\Omega^\circ X)(V) = \text{map}(S^V, X(V))$$

structure maps ~~&~~ by conj

$V \xrightarrow{\alpha} W$ in L gives

$$\alpha_*: \text{map}(S^V, X(V)) \rightarrow \text{map}(S^W, X(W))$$

is $f \mapsto$

$$S^W \cong S^V \wedge S^{W-\alpha(V)} \xrightarrow{f \wedge 1} X(V) \wedge S^{W-\alpha(V)}$$

$$\downarrow \tau_{V, W-\alpha(V)}$$

$$X(V \oplus (W-\alpha(V)))$$

$$\downarrow \cong \alpha$$

$$X(W)$$

$$\searrow \alpha_*(f)$$

For flat orthog spaces

unst. gl. eq \mapsto st. gl. eq.

Compute: $\pi_0^G(\Omega^\circ X)$ roughly $\frac{1}{2}$ $\&$ structure exists here

$\&$ other $\frac{1}{2}$ inherently stable

$$= \text{colim}_{V \subseteq \mathcal{U}_G} \pi_0((\Omega^\circ X)(V))^G$$

$$= \text{colim}_{V \subseteq \mathcal{U}_G} \pi_0(\text{map}^G(S^V, X(V))) = \text{colim}_{V \subseteq \mathcal{U}_G} [S^V, X(V)]^G = \pi_0^G(X)$$

This is an equality, not just an iso!

Cor: $\alpha: K \rightarrow G$ cont hom induces (an additive) $\pi_0^G(X) \rightarrow \pi_0^K(X)$ \forall orthog sp. X .

Restriction along all cont homs is something that is new here.

Next, transfer maps, as in Mike Hill's talks.

for $H \leq G$ get $\text{tr}_H^G: \pi_0^H(X) \rightarrow \pi_0^G(X)$ as usual

(N.B. no $[G:H]$ finite restriction) (Only the degree 0 transfer.)

$$\begin{array}{ccc} \text{Abel Sp} & \xleftarrow{\pi_*} & \text{Sp} \\ G\text{-Mackey} & \xleftarrow{\quad} & G\text{-Sp} \\ \mathcal{Y} \neq & \xleftarrow{\quad} & \mathcal{Y}H \\ = \text{global functors} & & \end{array}$$

Def: The Burnside category has objects cpt Lie groups & morphisms

$$A(G, K) = \text{Nat}(\pi_0^G, \pi_0^K)$$

as functors
 $\text{Sp} \rightarrow \text{Ab}$
 (equiv. $\text{Sp} \rightarrow \text{Set}$)

No other choice: use all the operations we have.

Obviously (pre-) additive

Def: A global functor is an additive functor $\mathcal{F}: A \rightarrow \text{Ab}$

Ex: $X \in \text{Sp}^0$ gives $(\pi_0 X)(G) = \pi_0^G(X)$

Fact: Every global functor \mathcal{F} has an Eilenberg-MacLane spectrum $\mathbb{H}\mathcal{F}$ s.t.

$$\pi_k(\mathbb{H}\mathcal{F}) = \begin{cases} \mathcal{F} & k=0 \\ 0 & \text{other} \end{cases}$$

Use t-structure. Global functors are the heart

$$\left. \begin{array}{l} \mathcal{Y}H_{\geq 0} = \{X \mid \pi_k^G X = 0 \text{ for } k < 0\} \\ \mathcal{Y}H_{\leq 0} = \{X \mid \pi_k^G X = 0 \text{ for } k > 0\} \end{array} \right\} \begin{array}{l} \text{a t-structure with} \\ \text{heart } \mathcal{Y}\mathcal{F}. \end{array}$$

General hearts may not have enough proj, inj but $\mathcal{Y}\mathcal{F}$ does (representables).

Fortunately we can describe $A(G, K)$ w/ a presentation, & see Mackey functors.

Thm: The group $A(G, K)$ is free abelian w/ basis the transformations $\text{tr}_L^K \circ \alpha^*$

where (L, α) range through all $K \times G$ conjugacy classes of

- subgroup $L \leq K$ with finite $W_K(L)$
- cont. hom $\alpha: L \rightarrow G$

(conj by $K \times G \rightarrow$ same op)

$$\pi_0^G(X) \xrightarrow{\alpha^*} \pi_0^L(X) \xrightarrow{\text{tr}_L^K} \pi_0^K(X).$$

$\pi_0^G : \mathcal{YH} \rightarrow \text{Ab}$ is rep'd by $\sum_+ B_{ge} G$

Then $\pi_0^K(\sum_+ B_{ge} G)$ is then calculated by the tom-Dieck splitting.

The double coset formula is the expression for composition in this basis.

If α is surjective, $\alpha: K \rightarrow G$, it is simple:

(*)
$$\begin{array}{ccc} \text{VI} & & \text{VI} \\ L & \xrightarrow{\alpha_L} & H \\ & & := \alpha^{-1}(G) \end{array} \quad \alpha^* \circ \text{tr}_H^G = \text{tr}_L^K \circ (\alpha_L)^*$$

Upshot: To specify a global functor M you must give

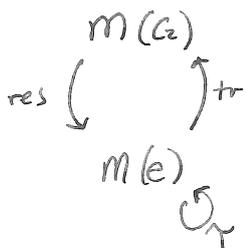
- the abelian group $M(G)$ for each G
- restrictions $\alpha^*: M(G) \rightarrow M(K)$
- transfers $\text{tr}_H^G: M(H) \rightarrow M(G)$

satisfying

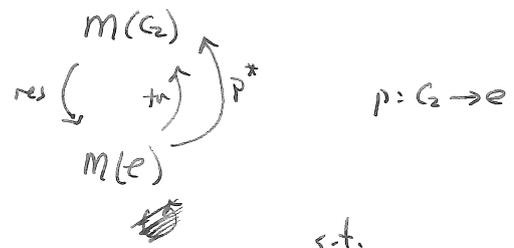
- restrictions are contravariantly functorial
- transfers are transitive
- (*) above
- double coset formula for $\text{res}_K^G \circ \text{tr}_H^G$
- $\text{tr}_H^G = 0$ if $|W_G H| = \infty$

Consider

G_2 -Mackey functors



Global functor at $G_2 \geq e$



s.t.

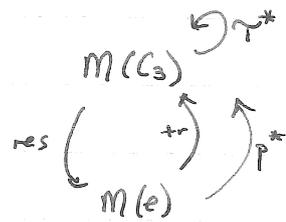
$\text{res} \circ p^* = 1$ (split epi: obstruct for a Mackey-functor to be part of a global functor)

~~res~~

$\text{res} \circ \text{tr} = 1 + \tau$

algebraic reflection of split homotopy type. \rightarrow

Question session: global functor at $C_3 \geq e$



$$\tau: C_3 \rightarrow C_3 \text{ by } \tau(x) = x^{-1}$$

Peter Webb calls these inflation functors. (the global functors?)

P. Symonds defines "regular global Mackey functors" via generators & relations.

\uparrow
 $\text{tr} = 0$ for
 ∞ Weyl gp

For finite groups $A_{\text{fin}} \cong A^{\text{comb}}$ defined as

$$A^{\text{comb}}(G, K) = \text{Gr gp of } G\text{-free finite } K\text{-}G\text{-bisets } {}_K S_G$$

$$[S] + [T] = [S \amalg T] \text{ and } [S][T] = [S \times_G T].$$

For $G \xleftarrow{\alpha} L \leq K$ you take $K \times_{\alpha, L} G = K \times G / (kl, g) \sim (k, \alpha(l)g)$

Then additive functors

$$A^{\text{comb}} \rightarrow A_b \text{ are called inflation functors (Webb)}$$

Algebraists have studied this a lot.

Dress also considered these.

Note $A(e, K) = A(K)$ Burnside ring

Choice of variance here fit well with $\pi_0^G \rightarrow \pi_0^K$. Algebraists often want to study the effect on contravariant functors, hence choose the other variance.

~~Ex: Rep'n Tomorrow~~