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Lecture 3 (2×45 min, 8 pages)

Start again w/ the stable situation.

Previous approaches:

Lewis-May-~~S~~ ∃ section encoding compatibility; change of group & universe functors

Greenlees & May: Completions of MU modules set up formalism to study norm maps, indexing on (G, V)

Böhmann: made more explicit, an equivalent category

Def: (formerly $\mathcal{I}_*$ prefunctors)

An orthogonal spectrum consists of

- based spaces X_n, $n \geq 0$
- $O(n)$ actions on each X_n,
- structure maps $\sigma_n : X_n \wedge S^1 \longrightarrow X_{n+1}$

such that $\forall n, m \geq 0$

$$X_n \wedge S^m \xrightarrow{\sigma_n \wedge 1} X_{n+1} \wedge S^{m-1} \longrightarrow X_{n+2} \wedge S^{m-2} \longrightarrow \cdots \longrightarrow X_{n+m}$$

is $O(n) \times O(m)$-equivariant.

Coordinate-free version requires Thom complexes over complements.

V an inner product space, $\dim V = n$, X orthog spectrum,

Def. $X(V) := L(\mathbb{R}^n, V)^+ \wedge_{O(n)} X_n$ as an $O(V)$ space.

Of course X_n in disguise but now coordinate free.

Generalized structure maps

$$X(V) \wedge S^W \longrightarrow X(V \oplus W), \quad O(V) \times O(W) \text{ equivariant.}$$

No G mentioned yet, but implicit via rep'ns:

If G cpt Lie and V is a G-rep'n then $X(V)$ has G action via $G \to O(V)$.

Note $X(\mathbb{R}^n) = X_n$ is fair statement since isomorphism is so canonical.

We get equivariant stable homotopy groups

Def: For $k \in \mathbb{Z}$

$$\pi_k^G(X) = \operatorname*{colim}_{V \subseteq \mathcal{U}_G} [S^{V+k}, X(V)] \qquad (k \geq 0;\ [S^V, X(V+k)] \text{ if } k<0)$$

poset of f.d. G sub rep'ns — complete

Formalism allowing pairs (G, universe) possible, but applications unclear.

These are morphisms in approp category: $[S^k, X]$.

Def: $X \xrightarrow{f} Y$ morphism of orthog. spectra is a <u>global equivalence</u> if

$$\pi_k^G(f) : \pi_k^G X \longrightarrow \pi_k^G Y$$

is an isomorphism $\forall k \in \mathbb{Z}$, $\forall G$.

Can restrict G to various subsets (e.g. finite) to get other notions of equivalence.

Def: The <u>global stable homotopy category</u> is $\mathcal{GH} = \mathcal{S}p\,[\text{gl.equiv.}^{-1}]$.

Thm: The global equivalences are part of a proper, cofibrantly generated, topological stable monoidal model category. The fibrant objects are the <u>global omega spectra</u> (Ω-spectra), i.e. the X s.t. ~~$\forall$ faithful~~ $\forall$ G and V, W G-rep'ns, V faithful, the map

$$\tilde{\sigma}_{V,W} : X(V) \longrightarrow \operatorname{map}(S^W, X(V \oplus W))$$

is a G wk equivalence.(*)

(*)
Key: Don't require a good homotopy type on non-faithful rep'ns. Meaningful values are only attained on faithful rep'ns. Requiring $\tilde{\sigma}_{V,W}$ wk equiv ...

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Can characterize cofib's via latching; Martin Stolz's cofib in flat model structure.

<u>Note</u>: For X a global Ω-spectrum, then $\forall G$, $\forall$ faithful G rep'n V, the G-space

$$\mathrm{map}(S^V, X(V))$$

is indep. of V up to G-wk equiv and a G-infinite loop space.

<u>Note</u>: Global Ω-spectrum $\Rightarrow$ non-eq Ω-spectrum.

<u>Note</u>: $H \leq G$, V G-faithful $\Rightarrow$ H-faithful so downward compatibility easy.
Upward depends on compatibility

<u>Note</u> Global Ω-spectra are intrinsically interesting w/o model structure.

Every global equivalence is a stable equivalence (non-eq.) So this is a finer notion. So $\exists$ forgetful functor, part of a recollement

$$\mathcal{GH}_+ \rightleftarrows \mathcal{GH} \overset{L}{\underset{R}{\rightleftarrows}} \mathcal{SH} = \mathcal{S}p[\mathrm{st.equiv}^{-1}]$$

$\{X : \pi_*^e X = 0\}$ (non-eq

(genlimning in Dansk)

L, R are fully faithful (need model cat structures, flat & proj)

$\mathcal{SH}$ is a localization of $\mathcal{GH}$; thinking of it as subcategory is possible via L and R, good for mapping out of or into, resp.

For a fixed G we have $\mathcal{S}p \longrightarrow G\text{-}\mathcal{S}p$ (orthog sp. w/ external G-action)

$X \longmapsto X$ w/ triv. G-action ~~[illegible]~~ $X\langle G\rangle$

no longer an equivalence on point set level (as above), instead fully faithful, but that vanishes upon localizing.

$X \xrightarrow{f} Y$ gl. eq. $\Rightarrow$ $f\langle G\rangle : X\langle G\rangle \longrightarrow Y\langle G\rangle$ is a G-stable equiv.
giving

$$U : \mathcal{GH} \rightleftarrows \mathrm{Ho}(G\text{-}\mathcal{S}p)$$

not a recollement. Has both adjoints but they're not fully faithful.

<u>Not</u> every G homotopy type is part of a global family. The global spectra are always split, for example.

For G, take all subgps & quotients & work relative to that family. Still not a part of a recollment, L&R not fully faithful.

Proof in 2 steps: first a level model structure. Then localize & it is no longer just about $O(n)$ but about G sitting in, in many ways.
Have to produce generators: take all closed subgps of $O(n)$, and that means all cpt Lie G. So just accepting coordinate-free approach, use all G, works best.

Global family: closed under iso, subgp (closed), & quotients.

All Cpt Lie
All Finite. $\mathcal{F}$
All Abelian cpt Lie Gps
All trivial Lie Gps

most used examples.

$$\begin{tikzcd} \mathcal{GH}_+^{\mathcal{F}} \arrow[r] & \mathcal{GH}_{\mathcal{F}} \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathcal{S}p \arrow[l, bend right, "L\mathcal{F}"'] \arrow[l, bend left, "R\mathcal{F}"] \end{tikzcd}$$

<u>Odd note</u>: Can mix families to get perverse examples.

Examples to come, but we need language for their properties.

One now: $\exists$ usual adjoint functor pair

$$\text{orthog. spaces} \underset{\Omega^\bullet}{\overset{\Sigma^\infty_+}{\rightleftarrows}} \mathcal{S}p$$

$$Y \in \text{orthog. spaces} \qquad X \in \mathcal{S}p$$

$(\Sigma^\infty_+ Y)(V) := Y(V)_+ \wedge S^V$

w/ diagonal $O(V)$-action

& structure map

$Y(V) \longrightarrow Y(V \oplus W)$

$S^V \wedge S^W \xrightarrow{\cong} S^{V \oplus W}$

For flat orthog spaces

unst. gl. eq $\longmapsto$ st. gl. eq.

$(\Omega^\bullet X)(V) = \mathrm{map}(S^V, X(V))$

structure maps ~~& acti~~ by conj

$V \xrightarrow{\alpha} W$ in L gives

$\alpha_* : \mathrm{map}(S^V, X(V)) \longrightarrow \mathrm{map}(S^W, X(W))$

$f \longmapsto$

is

$$\begin{tikzcd} S^W \cong_\alpha S^V \wedge S^{W-\alpha(V)} \arrow[r, "f \wedge 1"] \arrow[ddr, bend right, "\alpha_*(f)"'] & X(V) \wedge S^{W-\alpha(V)} \arrow[d, "\sigma_{V,W-\alpha(V)}"] \\ & X(V \oplus (W - \alpha(V))) \arrow[d, "\cong\ \alpha"] \\ & X(W) \end{tikzcd}$$

Compute: $\pi_0^G(\Omega^\bullet X)$ roughly ½ of structure exists here & other ½ inherently stable

$$= \operatorname*{colim}_{V \subseteq \mathcal{U}_G} \pi_0((\Omega^\bullet X)(V))^G$$

$$= \operatorname*{colim}_{V \subseteq \mathcal{U}_G} \pi_0(\mathrm{map}^G(S^V, X(V))) = \operatorname*{colim}_{V \subseteq \mathcal{U}_G} [S^V, X(V)]^G = \pi_0^G(X)$$

This is an equality, not just an iso!

Cor: $\alpha: K \to G$ cont hom induces (an additive) $\pi_0^G(X) \longrightarrow \pi_0^K(X)$ $\forall$ orthog sp. X.

Restriction along all cont homs is something that is new here.

Next, transfer maps, as in Mike Hill's talks.

for $H \leq G$ get $tr_H^G : \pi_0^H(X) \longrightarrow \pi_0^G(X)$ as usual

(N.B. no $[G:H]$ finite restriction) (Only the degree 0 transfer.)

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Abel Gps $\xleftarrow{\pi_*}$ Sp
G-Mackey $\longleftarrow$ G-Sp
GF $\longleftarrow$ GH
= global functors

Def: The Burnside category has objects cpt Lie groups & morphisms

$$A(G,K) = \mathrm{Nat}(\pi_0^G, \pi_0^K)$$

as functors $Sp \to Ab$ (equiv. $Sp \to Set$)

No other choice: use all the operations we have.
Obviously (pre-) additive

Def A global functor is an additive functor $F: A \longrightarrow Ab$

Ex: $X \in Sp^0$ gives $(\underline{\pi}_0 X)(G) = \pi_0^G(X)$

Fact: Every global functor F has an Eilenberg-MacLane spectrum HF s.t.

$$\underline{\pi}_k(HF) = \begin{cases} F & k=0 \\ 0 & \text{other} \end{cases}$$

Use t-structure. Global functors are the heart

$$\left.\begin{aligned} GH_{\geq 0} &= \{X \mid \pi_k^G X = 0 \text{ for } k<0\} \\ GH_{\leq 0} &= \{X \mid \pi_k^G X = 0 \text{ for } k>0\} \end{aligned}\right\} \text{ a t-structure with heart } GF.$$

General hearts may not have enough proj, inj but GF does (representables).

Fortunately we can describe $A(G,K)$ w/ a presentation, & see Mackey functors.

Thm: The group $A(G,K)$ is free abelian w/ basis the transformations

$$tr_L^K \circ \alpha^*$$

where (L,α) range through all $K\times G$ conjugacy classes of
- subgroup $L \leq K$ with finite $W_K(L)$
- cont. hom $\alpha: L \to G$

$$\pi_0^G(X) \xrightarrow{\alpha^*} \pi_0^L(X) \xrightarrow{tr_L^K} \pi_0^K(X).$$

(conj by $K \times G \to$ same op)

$\pi_0^G : \mathcal{G}H \to Ab$ is rep'd by $\Sigma_+^\infty B_{gl}$

Then $\pi_0^K(\Sigma_+^\infty B_{gl} G)$ is then calculated by the tom-Dieck splitting.

The double coset formula is the expression for composition in this basis.

If α is surjective, $\alpha : K \to G$, it is simple:

(*) $\begin{array}{ccc} \cup\!\shortmid & & \cup\!\shortmid \\ L & \xrightarrow{\alpha|_L} & H \\ {:=}\alpha^{-1}(G) & & \end{array}$ $\qquad \alpha^* \circ tr_H^G = tr_L^K \circ (\alpha|_L)^*$

<u>Upshot</u>: To specify a global functor M you must give
- the abelian group $M(G)$ for each G
- restrictions $\alpha^* : M(G) \to M(K)$
- transfers $tr_H^G : M(H) \to M(G)$

satisfying
- restrictions are contravariantly functorial
- transfers are transitive
- (*) above
- double coset formula for $res_K^G \circ tr_H^G$
- $tr_H^G = 0$ if $|W_G H| = \infty$

Consider
C$_2$-Mackey functors

$$\begin{tikzcd} M(C_2) \arrow[d, bend right, "res"'] \\ M(e) \arrow[u, bend right, "tr"'] \arrow[loop below, "\tau"] \end{tikzcd}$$

~~res o tr~~

$res \circ tr = 1 + \tau$

Global functor at $C_2 \supseteq e$

$$\begin{tikzcd} M(C_2) \arrow[d, bend right, "res"'] \\ M(e) \arrow[u, bend right, "tr"'] \arrow[u, bend right=70, "p^*"'] \end{tikzcd}$$

$p : C_2 \to e$

s.t.

$res \circ p^* = 1$ (split epi: obstruc for a Mackey-functor to be part of a global functor)

algebraic reflection of split homotopy type.

Question session: global functor at $C_3 \supseteq e$

$$\begin{tikzcd} M(C_3) \arrow[loop right, "\tau^*"] \arrow[d, bend right, "res"'] \\ M(e) \arrow[u, bend right, "tr"'] \arrow[u, bend right=60, "p^*"'] \end{tikzcd}$$

$\tau : C_3 \to C_3$ by $\tau(x) = x^{-1}$

Peter Webb calls these inflation functors. (the global functors?)

P. Symonds defines "regular global Mackey functors" via generators & relations.

↑
tr = 0 for
∞ Weyl gp

For finite groups $A_{fin} \cong A^{comb}$ defined as

$$A^{comb}(G,K) = \text{Gr gp of } G\text{-free finite } K\text{-}G\text{-bisets } {}_K S_G$$

$[S] + [T] = [S \amalg T]$ and $[S][T] = [S \times_G T]$.

For $G \xleftarrow{\alpha} L \leq K$ you take $K \times_{\alpha, L} G = K \times G / (kl, g) \sim (k, \alpha(l)g)$

Then additive functors

$$A^{comb} \longrightarrow Ab \text{ are called inflation functors (Webb)}$$

Algebraists have studied this a lot.

Dress also considered these.

Note $A(e,K) = A(K)$ Burnside ring

Choice of variance here fit well with $\pi_0^G \longrightarrow \pi_0^K$. Algebraists often want to study the effect on contravariant functors, hence choose the other variance.

~~Ex: Rep'n~~ Tomorrow