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Lecture 2 (2×45 min, 10 pages)

Still unstable global homotopy theory.

L = cat of f.d. real i.p. spaces w/ lin. isom. embeddings

Orthogonal spaces are functors $Y: L \to \mathcal{T}op$

G will always be cpt. Lie, $\mathcal{U}_G$ a complete G universe

Underlying G-space $Y(\mathcal{U}_G) = \operatorname{colim}_{V \subseteq \mathcal{U}_G} Y(V)$

i.e. when Y, Z closed

$Y \xrightarrow{f} Z$ is a global equivalence $\Leftrightarrow$ $f(\mathcal{U}_G): Y(\mathcal{U}_G) \to Z(\mathcal{U}_G)$ is a G wk equiv

$\pi_0^G(Y) = \operatorname{colim}_{V \subseteq \mathcal{U}_G} \pi_0(Y(V)^G) \simeq \pi_0(Y(\mathcal{U}_G)^G)$

$K \xrightarrow{\alpha} G$ cont hom $\Rightarrow \alpha^*: \pi_0^G(Y) \to \pi_0^K(Y)$

(Note this sort of restriction along arbitrary homs, not just subgps, applies in this global setting; rarely available "one gp at a time")

$\underline{\pi_0}(Y) = \{\pi_0^G(Y)\}: Rep^{op} \to \mathcal{S}ets$

obj = cpt Lie
mor = conj cl. of hom's

orthog monoid spaces have $\mu_{V,W}: Y(V) \times Y(W) \to Y(V \oplus W)$

$1 \in Y(0)$

From this $\exists$ exterior product on $\underline{\pi_0}(Y)$:

$\mu_{\mathcal{U}_G, \mathcal{U}_K}: Y(\mathcal{U}_G) \times Y(\mathcal{U}_K) \to Y(\mathcal{U}_G \oplus \mathcal{U}_K)$

a universe for $G \times K$, typically not complete

$\exists$ contractible space of embeddings $\mathcal{U}_G \oplus \mathcal{U}_K \xhookrightarrow{\phi} \mathcal{U}_{G \times K}$

so

$$\pi_0^G(Y) \times \pi_0^K(Y) = \pi_0(Y(\mathcal{U}_G)^G) \times \pi_0(Y(\mathcal{U}_K)^K)$$
$$\cong \pi_0((Y(\mathcal{U}_G) \times Y(\mathcal{U}_K))^{G \times K}) \xrightarrow{\mu_*} \pi_0(Y(\mathcal{U}_G \oplus \mathcal{U}_K)^{G \times K})$$
$$\xrightarrow{\phi_*} \pi_0(Y(\mathcal{U}_{G \times K})^{G \times K}) = \pi_0^{G \times K}(Y)$$

Internalize via $G \xrightarrow{\Delta} G \times G$ as usual.

So

$\pi_0^G(Y)$ is a monoid, natural in G, so $\underline{\pi_0}: Rep^{op} \to \mathcal{M}onoids$.

Y comm $\Rightarrow \pi_0^G(Y)$ are comm monoids

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Strict comm has an extra bonus (prev. structure needed only homotopy commutative $\mu_{v,w}$). These are

power operations / norm maps (mult. setting)
transfers (add. setting)

Iterated mult. maps

$$\underbrace{Y(V)\times\cdots Y(V)}_{m} \xrightarrow{\mu_{V,V,\dots,V}} Y(V\oplus\cdots\oplus V) = Y(V^m)$$

If V is a G-rep'n then $\overset{\text{this is}}{\Sigma_m\wr G}$ - equivariant. (with operads, how would one do this?)

Choose $\Sigma_m\wr G$ - equivariant lin. isom. embedding $\mathcal{U}_G^m \overset{\varphi}{\hookrightarrow} \mathcal{U}_{\Sigma_m\wr G}$
($\mathcal{U}_G^m$ is a universe, not complete in general)

$$P^m : \pi_0^G(Y) = \pi_0(Y(\mathcal{U}_G)^G) \xrightarrow{\Delta_*} \pi_0((Y(\mathcal{U}_G)^m)^{\Sigma_m\wr G})$$

$$x \longmapsto (x,\dots,x)$$

$$\xrightarrow{\mu_*} \pi_0(Y(\mathcal{U}_G^m)^{\Sigma_m\wr G})$$

$$\xrightarrow{\varphi_*} \pi_0(Y(\mathcal{U}_{\Sigma_m\wr G})^{\Sigma_m\wr G}) = \pi_0^{\Sigma_m\wr G}(Y)$$

<u>Properties</u>: of

$$P^m : \pi_0^G(Y) \longrightarrow \pi_0^{\Sigma_m\wr G}(Y) \qquad Y \text{ a c.o.m.s.}$$

(think of $x\mapsto x^m$ in a comm monoid)

(i) $P^m(1) = 1$ (different 1's of course)
(ii) $P^1 = \mathrm{Id}$ (using $\Sigma_1\wr G \cong G$)
(iii) $K \xrightarrow{\alpha} G$ cont. hom $\Rightarrow P^m(\alpha^*(x)) = (\Sigma_m\wr\alpha)^*(P^m(x))$ in $\pi_0^{\Sigma_m\wr K}(Y)$
(iv) $P^m(x\cdot y) = P^m(x)\cdot P^m(y)$
(v) $P^i(x)\cdot P^{m-i}(x) = \Phi^*(P^m(x))$ in $\pi_0^{\Sigma_i\wr G\times\Sigma_{m-i}\wr G}$
using $\Phi: \Sigma_i\wr G\times\Sigma_{m-i}\wr G \hookrightarrow \Sigma_m\wr G$ (block sum)
(vi) $P^k(P^m(x)) = (\Phi')^* P^{km}(x)$
in $\pi_0^{\Sigma_k\wr(\Sigma_m\wr G)}(Y)$
using $\Phi' : \Sigma_k\wr(\Sigma_m\wr G) \hookrightarrow \Sigma_{mk}\wr G$

We'll see this in yesterday's examples.

<u>Note:</u> $\pi_0^G(Y) \xrightarrow{P^m} \pi_0^{\Sigma_m \wr G}(Y) \xrightarrow{\Delta^*} \pi_0^G(Y)$ $\quad G \xrightarrow{\Delta} G^m \subset \Sigma_m \wr G$

is $\quad x \longmapsto x^m$

The algebra of all natural operations is generated by these.
No additive structure yet. Power operations $\leftrightarrow$ Norm maps.
Note importance of global structure: $G \rightsquigarrow \Sigma_m \wr G$

<u>Norm maps</u> (first showed up in gp cohom; Greenlees & May; Hill-Hopkins-Ravenel)

Let $H \leq G$ (closed subgp) of finite index $m = [G:H]$.
Let $\langle G:H \rangle$ be the set of all coset rep's, i.e. subset of G^m containing those $(g_1, \dots, g_m)$ s.t. $G = \bigcup_{i=1}^m g_i H$.

$G \times \langle G:H \rangle \longrightarrow \langle G:H \rangle$ diagonally $\gamma \cdot (g_1, \dots, g_m) = (\gamma g_1, \dots, \gamma g_m)$

$\langle G:H \rangle \times (\Sigma_m \wr H) \longrightarrow \langle G:H \rangle \quad (g_1, \dots, g_m) \cdot (\sigma; h_1, \dots, h_m) = (g_{\sigma(1)} h_1, \dots, g_{\sigma(m)} h_m)$

and this is a free & transitive action

Choose one $\bar{g} \in \langle G:H \rangle$, $\bar{g} = (g_1, \dots, g_m)$ and define a homomorphism

$$\Phi : G \longrightarrow \Sigma_m \wr H \quad \text{by} \quad \gamma \cdot \bar{g} = \bar{g} \cdot \Phi(\gamma)$$

Φ is independent of $\bar{g}$ up to conjugation by $\Sigma_m \wr H$

<u>Def:</u> (Evens; Greenlees & May; Hill, Hopkins & Ravenel)

$$N_H^G = \Phi^* \circ P^m : \pi_0^H(Y) \longrightarrow \pi_0^G(Y)$$

$$\begin{tikzcd} \pi_0^H(Y) \arrow[dr, "P^m"'] & & \pi_0^G(Y) \\ & \pi_0^{\Sigma_m \wr H}(Y) \arrow[ur, "\Phi^*"'] & \end{tikzcd}$$

is the <u>norm map</u> for $H \leq G$, $m = [G:H] < \infty$.

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Properties: (1) $N_H^G(1) = 1$ (different 1's of course)

$N_G^G(x) = x$

(2) $N_H^G \circ N_K^H = N_K^G$ if $K \leq H \leq G$, $[G:K] < \infty$

(3) $N_H^G(x \cdot y) = N_H^G(x) \cdot N_H^G(y)$

(4) $K \leq G$, $H \leq G$ $\Rightarrow$

$$\mathrm{res}_K^G \circ N_H^G = \prod_{[g] \in K\backslash G/H} N^K_{K \cap {}^gH}\, c_g^* \circ \mathrm{res}^H_{K^g \cap H}$$

—— BREAK ——

For Y a c.o.m.s. (comm orthog monoid space) (global model for E_∞ spaces)

$\pi_0(Y)$ is a global power monoid:

- $\pi_0^G(Y)$ comm monoid
- $\alpha^* : \pi_0^G(Y) \longrightarrow \pi_0^K(Y)$
- $P^m : \pi_0^G(Y) \longrightarrow \pi_0^{\Sigma_m \wr G}(Y)$ (equiv. Norm maps)

Ex: Gr = additive Grassmanian $Gr(V) = \coprod_{n \geq 0} Gr_n(V)$

write $Gr^{[n]}(V) = Gr_n(V)$

with $\mu_{V,W}(L, L') = L \oplus L'$

Claim: $\pi_0^G(Gr) = RO^+(G)$ monoid of iso cl. of G rep'ns

Proof: $Gr^{[n]} = B_{gl}O(n)$ since $L(\mathbb{R}^n, V)/O(n) \cong Gr_n(V)$

$\phi\, O(n) \longmapsto \phi(\mathbb{R}^n)$

and

$\pi_0^K B_{gl} O(n) = \mathrm{Rep}(K, O(n)) =$ Iso. cl. of n dim'l rep'ns.

Explicit iso

$$\pi_0(Gr(V)^G) \longrightarrow RO^+(G)$$

$$W \in Gr(V)^G \longmapsto [W]$$

means it inherits a G-action

well def on path components since rep'ns are discrete.

This iso is surj: have a G-embedding $W \overset{\phi}{\hookrightarrow} \mathcal{U}_G$,
$W' = \phi(W) \cong W$

$$\pi_0(Y(W')^G) \longrightarrow \underset{V \subseteq \mathcal{U}_G}{\operatorname{colim}} \pi_0(Y(V)^G) = \pi_0^G(Gr)$$

$Y = Gr$ $\qquad$ $\underset{W'}{\in}$

and Restriction = restriction

$$P^m \leftrightarrow \underset{RO^+(G)}{[V]} \longmapsto V^m \in RO^+(\Sigma_m \wr G)$$

where $[V] \in RO^+(G)$.

$$N_H^G \leftrightarrow tr : [V] \longmapsto [\operatorname{map}^H(G, V)] = [\mathbb{R}G \otimes_{\mathbb{R}H} V] \in RO^+(G)$$

where $[V] \in RO^+(H)$.

<u>Ex</u>: $BOP(V) = \coprod_{m \geq 0} Gr_m(V^2)$ $\qquad$ $\mu_{V,W}(L, L') = \kappa_{V,W}(L \oplus L')$

$$BOP(V) \longrightarrow BOP(V \oplus W)$$
$$V^2 \supseteq L \longmapsto \kappa_{V,W}(L \oplus (W \oplus 0))$$

where $\kappa_{V,W}(L \oplus (W \oplus 0)) \subseteq (V \oplus W)^2 \cong V^2 \oplus W^2$

$$\pi_0^G BOP = RO(G)$$
$$\pi_0^G BOP = \underset{V \subseteq \mathcal{U}_G}{\operatorname{colim}} \pi_0(BOP(V)^G)$$

For V a G-rep'n $\qquad$ $W \in (BOP(V))^G \iff G$ inv subsp of V^2

$$[U] \longmapsto [U] - [V] \in RO(G)$$

because of different stabilization, these are the compatible seq's in colim.
That's why you get RO, not just RO^+.
Surj ↴

V, W G-rep'ns, so $[W]-[V] \in RO(G)$

$$= [W \oplus W] - [V \oplus W]$$

and

$$W \oplus W \underset{G}{\hookrightarrow} (V \oplus W)^2$$

Ex $G = e$ $\quad \pi_0(BOP(V)) = \pi_0\left(\coprod_{m \geq 0} Gr_m(V^2)\right) = \{-m, \dots, 0, \dots, m\}$ $\qquad m = \dim V$

$$Gr_{i+\dim V} \leftrightarrow i$$

[Re group completion. This is additive. $\exists$ homotopical group completion which is alg. gp. completion on π_0. Also $Gr \xrightarrow{i} BOP$ is global gp completion.

at V:

$$\begin{array}{ccc} Gr(V) & \longrightarrow & BOP(V) \\ \| & & \| \\ \coprod_{n \geq 0} Gr_n(V) & & \coprod_{m \geq 0} Gr_m(V^2) \end{array}$$

induces $RO^+(G) \to RO(G)$ on π_0^G

$$V \geq L \longmapsto L \oplus V \leq V^2$$

Claim: i is gp completion of c.o.m.s. (derived mapping space to gp complete c.o.m.s. agree). So deloopings agree.

General gp completion: R a c.o.m.s, define R^+ as a pushout

$$\begin{tikzcd} R \arrow[r, "\Delta"] \arrow[d] \arrow[dr, phantom, "\text{ho p.o.}"] & R \times R \arrow[d] & R \arrow[l, "\binom{1}{0}"'] \arrow[dl, "\text{gp compl}"] \\ * \arrow[r] & R^+ & \end{tikzcd}$$

in comm orthog mon spaces.

(Same for monoids (not sets))

(Segal's also Quillen's $S^{-1}S$)

Four Orthogonal Spaces, different globally, all w/ underlying space BO
(2 w/ products, 2 w/o)

(1) constant BO, $\underline{BO}$ (stupid one) $\quad \underline{BO}(\mathcal{U}_G) = BO$ w/ triv G-action

so $\pi_0^G(\underline{BO}) = *$

(2) Bar construction $B\mathbf{O} = B\mathbf{O}$, where $\mathbf{O}(V) = O(V)$, structure map conj.

$(\mathbf{O}(\mathcal{U}_G))^G$ = G-equivariant orthog

$(B\mathbf{O})(V) = BO(V)$ and ~~orientations~~

$\pi_0^G BO = *$

(3) hocolim $B_{gl}O(n)$ Non-equiv. $\mathbf{O} = \operatorname{colim} O(n)$

$BO = \operatorname{hocolim} O(n)$

$B_{gl}O(n) \longrightarrow B_{gl}O(n+1)$

via $O(n) \longrightarrow O(n+1)$

$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$

$\pi_0^G \operatorname{hocolim} B_{gl}O(n) = \operatorname{colim}_n \pi_0^G B_{gl}O(n)$

$= \operatorname{colim}_n \Big(\underset{\substack{\text{iso cl} \\ \text{of } n \\ \text{dim} \\ \text{rep'n}}}{\underset{\parallel}{RO_n(G)}} \xrightarrow{-\oplus\mathbb{R}} RO_{n+1}(G) \to \cdots \Big)$

stabilizing by adding inverses to trivial rep'ns only.

Mult?

(4) $B\mathbf{O} = BOP^{[0]}$, i.e. $B\mathbf{O}(V) = Gr_{\dim V}(V^2)$

$\pi_0^G B\mathbf{O} = IO(G) \subset \underset{\underset{\pi_0^G BOP}{\parallel}}{RO(G)}$ aug. ideal

so BOP is the better obj.

<u>Bott map</u> globally (def of map, not a proof)

goes back to McDuff, made more precise by Aguilar-Prieto, finally Behrens, almost what is here.

Switch to complex version: $Gr^{\mathbb{C}}(V) = \coprod_{n\geq 0} Gr_n^{\mathbb{C}}(V_{\mathbb{C}})$

and

$$\mathbf{U}(V) = U(V_{\mathbb{C}})$$

$\beta : Gr^{\mathbb{C}} \longrightarrow \Omega\mathbf{U}$ globally delopable

objectwise loops based at identity

w/ ptwise mult of loops.

Define

$sa(V)$ = self adjoint endo of $V_{\mathbb{C}}$ $= \{ f \in End_{\mathbb{C}}(V_{\mathbb{C}}) \mid \langle x, fy\rangle = \langle fx, y\rangle \}$

a contractible (by linear homotopy to 0) real vector-space

$$\mu_{V,W} : sa(V) \times sa(W) \longrightarrow sa(V \oplus W)$$
$$f, g \longmapsto f \oplus g$$

unit 0

Lie theoretic tangent space

$$\exp : sa(V_{\mathbb{C}}) \longrightarrow U(V_{\mathbb{C}})$$
$$A \longmapsto e^{2\pi i A} = \sum_{n\geq 0} \frac{(2\pi i)^n}{n!} A^n$$

A morphism of coms!

Now also $P : Gr^{\mathbb{C}} \longrightarrow sa$

at V

$$\coprod_{n\geq 0} Gr_n^{\mathbb{C}}(V_{\mathbb{C}}) \longrightarrow sa(V_{\mathbb{C}})$$
$$L \longmapsto \text{orthog. proj onto } L = P_L$$

Prop
$$\begin{tikzcd} Gr^{\mathbb{C}} \arrow[r, "P"] \arrow[d] & sa \arrow[d] \\ * \arrow[r] & \mathbf{U} \end{tikzcd}$$
is a homotopy pushout of coms (Global Bott periodicity)

Note exp $P_L = 1$ since eigenvalues are $e^{2\pi i 0} = 1$ and $e^{2\pi i 1} = 1$

Pullback would be pointwise loops, group complete, but $Gr^{\mathbb{C}}$ is not.

Almost additive setting ⇝ pushout nearly the same.

Results in

Cor: $BOP \xrightarrow{\sim} \Omega U$

Real case: use Atiyah's Real vector spaces.

Question session:

not. trans.

~~Prop~~ Thm: Every natural operation $\pi_0^G \longrightarrow \pi_0^K$ of functors Orthog Spaces $\to$ Sets is of the form α^* for a unique conj. cl. of cont. hom $\alpha : K \to G$

Thm: Every nat trans $\pi_0^G \to \pi_0^K$ of functors C.oms $\longrightarrow$ Sets is of the form $\alpha^* \circ p^m$ for a unique $m \geq 0$ and ~~α as above~~ conj. cl. of cont. hom $\alpha : K \longrightarrow \Sigma_m \wr G$.

Similar thm in case of spectra.

Proof by computing values on representing space.

Algebras over a multi-sorted theory w/ objects cpt Lie gps, but that probably doesn't add much.

A an abelian cpt Lie gp. Then $B_{gl}A$ has a model as a coms.

e.g. $B_{gl}U(1)$, oops, $B_{gl}O(1) = P$, $P(V) = P(\mathrm{Sym}(V))$

$\pi_0^K B_{gl} G = \mathrm{Rep}(K, G)$ but $G = A$ abelian gives ⇝

$\pi_0^K B_{gl}(A) = \mathrm{Rep}(K,A)$ = homs since conj triv
$= \mathrm{Hom}(K,A)$ + is sum of homs &

$$P^m : \mathrm{Hom}(K,A) \longrightarrow \mathrm{Hom}(\Sigma_m \wr K, A)$$

$$f \longmapsto \quad \Sigma_m \wr K \xrightarrow[\Sigma_m \wr f]{} \Sigma_m \wr A \xrightarrow{\text{sum}} A$$

$$(\sigma; a_1, \ldots, a_m) \longmapsto \sum_1^m a_i$$

Evens norm? Need cohomology; Borel theory?
or $K(A,n) \longrightarrow$??