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Monday Lectures (2×45 min, 10 pages)

Global Stable Homotopy Theory (book on home page & conf. page)

All groups act simultaneously and compatibly.

Global = actions of "all" groups (all = cpt Lie, or finite, or ??) (use cpt Lie at points)

2 things: (1) rigorous formalism and (2) examples

Mon, Tues: unstable world: space on which all act

Wed, Thu: stable

Fri: ? Questions will drive it.

Unstable: An orthogonal space, ($\mathcal{I}$-space, $\mathcal{L}$ space)

L = cat of f.d. (real) i.p. space w/ linear isom. embeddings

Topological cat: $L(V, W) = \emptyset$ or $O(W)$ acts transitively so it is $O(W)/O(W-V)$

$W - V$ = orthog. complement. (Space = cpt gen wk Hausdorff)

Def: An orthogonal space is a cont. functor $X : L \to Top$

studied by: Boardman, Boardman & Vogt, May et al, Lind, Sagave-Schlichtkrull (at least)

These have been studied as a model of spaces, primarily. Easy def of E_∞ objects.

Def: $Y \xrightarrow{f} Z$ is a wk. equiv if $\underset{n \geq 0}{Tel}\, f(\mathbb{R}^n) : Tel\, Y(\mathbb{R}^n) \longrightarrow Tel(X(\mathbb{R}^n)$ is a wk equiv.

$$\begin{tikzcd} \text{orthog spaces} \arrow[r, shift left, "Tel_n"] & Top \arrow[l, shift left, "const"] \end{tikzcd}$$

induce $Ho(\text{orthog space}) \cong Ho(Top)$

Comm Monoid models E_∞ space

Comm Monoid too strict

$\nwarrow$ Advantage

(Note: orthog spectra w.r.t.

New: A finer notion of equivalence.

Let G be a cpt Lie group., V a G-representation (usually finite, always orthog.)

Y an orthog space $\Rightarrow$ $Y(V)$ is a G-space

Slogan: $Y \xrightarrow{f} Z$ is a global equivalence if $\forall$ G cpt Lie $\rightarrow$

$\swarrow$ B.K. hocolim

$$\operatorname{hocolim}_V Y(V) \longrightarrow \operatorname{hocolim}_V Z(V) \quad \text{is a } G\text{-wk equiv}$$

(Can make this precise, but formal def is advantageous (& equiv.)

<u>Def</u>: $Y \xrightarrow{f} Z$ is a <u>global equivalence</u> if $\forall G$ cpt Lie, V G-rep'n and comm. square

$$\begin{tikzcd}
S^{k-1} \arrow[r] \arrow[d, hook] & Y(V)^G \arrow[d] \\
D^k \arrow[r] & Z(V)^G
\end{tikzcd}$$

$\exists$ G-rep'n W and diagram

$$\begin{tikzcd}
S^{k-1} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowright"] & Y(V)^G \arrow[r] & Y(V \oplus W)^G \arrow[d, "f(V \oplus W)^G"] \\
D^k \arrow[r] \arrow[urr, dashed] & Z(V)^G \arrow[r] \arrow[ur, phantom, "\overset{h}{\circlearrowright}"] & Z(V \oplus W)^G
\end{tikzcd}$$

(if $W = 0$ just wk equiv)

$\exists$ model structure w/ this, reassuring us this may be interesting.

<u>Thm</u>: The category of orthogonal spaces has a <u>global model structure</u> with
- weak equivs = global equivalences
- cof = flat cofibration, $A \xrightarrow{i} B$ $\quad \operatorname{Lim} B \underset{\operatorname{Lim} A}{\cup} A(\mathbb{R}^m) \longrightarrow B(\mathbb{R}^m)$ is an $O(m)$-cof = retract of equiv CW complex

Can characterize fibrant obj: Y is <u>globally fibrant</u> if $\forall G, V$ s.t. V is faithful, $\forall$ G rep'n W, $Y(V) \longrightarrow Y(V \oplus W)$ is a <u>G-wk. equiv.</u>
(Essentially constant, starting at first reasonable space) $\curvearrowright$ i.e. $\forall H \leq G$ $f(V)^H$ is a wk equiv.

<u>Ex</u>: $A \in \mathcal{T}op$, $\underline{A}$ = const. orthog space, $\underline{A}(V) = A$ is $\underset{\text{globally}}{\smile}$ fibrant
More examples from K-theory.

Criterion for global equiv.

Def: An orthog. space Y is closed if all $Y(\alpha): Y(V) \to Y(W)$ are closed embeddings, $\forall\ V \overset{\alpha}{\hookrightarrow} W$.

Reminder: A complete G-universe is a G-rep'n (Not f.d.) $\mathcal{U}_G \cong \mathbb{R}^\infty$ s.t. every f.d. G-rep'n occurs countably often in $\mathcal{U}_G$.

G-finite has $\mathcal{U}_G = \bigoplus_{\mathbb{N}} \rho_G$, $\rho_G = \mathbb{R}[G]$ reg. rep'n.

G arbitrary similarly with $\rho_G = \bigoplus_{\substack{\text{iso classes}\\ \text{of irreps } \lambda}} \lambda$

Isom contractible $\Rightarrow$ as unique as could be expected.

Def: The underlying G-space of an orthogonal space Y is

$$Y(\mathcal{U}_G) := \operatorname*{colim}_{\substack{V \subseteq \mathcal{U}_G \\ \dim V < \infty}} Y(V) \qquad \text{(a G-space)}$$

(Only good for closed Y, of course.)

Prop: Y, Z closed orthog. spaces. TFAE for $Y \xrightarrow{f} Z$

(i) f is a global equivalence

(ii) $\forall G$ $f(\mathcal{U}_G): Y(\mathcal{U}_G) \to Z(\mathcal{U}_G)$ is a G-wk-equiv.

(Equivalently, $Y(\mathcal{U}_G)^G \to Z(\mathcal{U}_G)^G$ is wk equiv since $\forall G$)

(Note: cofib $\Rightarrow$ closed)

Examples: (0) stupid example (constant)

~~(1)~~ $A \xrightarrow{g} B$ is a wk equiv $\Leftrightarrow$ $\underline{g}: \underline{A} \to \underline{B}$ is a global equiv.

(1) Global classifying spaces (!) G cpt Lie. Make a choice of faithful G-rep'n V, (but this will vanish, i.e. be irrelevant). Then $\rightsquigarrow$

$$B_{gl}G\,(W) := L(V,W)/G$$

Then $B_{gl}G\,(\mathbb{R}^\infty) = L(V,\mathbb{R}^\infty)/G = \text{"}EG\text{"}/G = BG$

$\uparrow$ G CW, $EG \simeq *$,

(Better than constant BG, bar construction, etc.)

Let K be another cpt Lie group. Then

$$B_{gl}G\,(\mathcal{U}_K) = L(V,\mathcal{U}_K)/G$$

$\leftarrow$ a $K\times G^{op}$ space, universal for the family of those $\Gamma \subseteq K\times G^{op}$ s.t. $\Gamma \cap 1\times G^{op} = \{(1,1)\}$.

(ie $\quad L(V,\mathcal{U}_K)^\Gamma \simeq \begin{cases} \emptyset & \Gamma\cap 1\times G^{op} \neq \{(1,1)\} \\ \simeq * & \Gamma\cap 1\times G^{op} = \{(1,1)\} \end{cases}$

This universality shows choice of V didn't matter.

<u>Hence</u>, $\quad B_{gl}G\,(\mathcal{U}_K) = L(V,\mathcal{U}_K)/G \quad$ is the classifying space for principal G-bundles over K-spaces.

(just as $B_{gl}G\,(\mathbb{R}^\infty)$ was such for trivial K-spaces)

— BREAK —

G cpt Lie, V faithful G-rep, $\quad B_{gl}G\,(W) = L(V,W)/G$

<u>Lemma</u>: The global homotopy type of $B_{gl}G$ is indep. of V.

<u>Pf</u>: Let $\bar V$ be another faithful G-rep. Then $V\oplus\bar V$ is a third and we have

$$\cancel{L(V)/G} \longleftarrow \cancel{L(V\oplus\bar V)/G} \;\cancel{\longrightarrow}\; \cancel{L(\bar V)/G}$$

$$L(V,W)/G \longleftarrow L(V\oplus\bar V,W)/G \longrightarrow L(\bar V,W)/G \qquad \text{are global equiv.}$$

since, for K cpt Lie

$$L(V\oplus\bar V,\mathcal{U}_K) \longrightarrow L(V,\mathcal{U}_K) \quad \text{is a } K\times G^{op} \quad \text{between universal}$$

spaces for the same family of subgroups of $K \times G^{op}$
$\Rightarrow K \times G^{op}$ - homotopy equiv $\Rightarrow L(V \oplus \bar{V}, \mathcal{U}_K)/G \longrightarrow L(V, \mathcal{U}_K)/G$ is a K-homotopy equivalence. //

Ex: $G = C_2$, reg rep'n faithfull, but sign σ on $\mathbb{R}$ is smaller.

$$B_{gl}C_2(W) = L(\sigma, W)/C_2 = S(W)/C_2 = P(W)$$

$B_{gl}C_2(\mathcal{U}_K) = P(\mathcal{U}_K)$ classifying space for real line bundles / K-spaces

Other examples will need

Def: an orthogonal monoid space is

(i) lax monoidal functor $Y : (L, \oplus) \longrightarrow (\mathcal{T}op, \times)$

unit: 0 $\quad *$

i.e.

(ii) an orthogonal space Y w/ maps $\mu_{V,W} : Y(V) \times Y(W) \longrightarrow Y(V \oplus W)$

w/ $1 \in Y(0)$

w/ $\mu_{V,W}$ $O(V) \times O(W)$ equivariant, and satisfying associativity.

i.e.

(iii) a monoid in orthog spaces w.r.t. $\boxtimes$, a Day type convolution product.

An orthog monoid space is commutative if

(i) lax symmetric monoidal

(ii)
$$\begin{tikzcd} Y(V) \times Y(W) \arrow[r, "\mu_{V,W}"'] \arrow[d] & Y(V \oplus W) \arrow[d] \\ Y(W) \times Y(V) \arrow[r, "\mu_{W,V}"'] & Y(W \oplus V) \end{tikzcd}$$
commutes

(iii) comm monoid.

Preview: Comm orthog monoid spaces can be globally delooped.
(Caveats for non finite G. Stay tuned.)

Ex $\mathbb{O}$: $\mathbb{O}(V) := O(V)$ orthog gp

Structure maps by conjugation: $V \xhookrightarrow[\alpha]{} W$ in L gives

$$\mathbb{O}(\alpha) : O(V) \longrightarrow O(W)$$

$$A \longmapsto \left\{ W = \alpha(V) \oplus \alpha(V)^{\perp} \xrightarrow{\alpha L \alpha^{-1} \oplus 1} \alpha(V) \oplus \alpha(V)^{\perp} \right\}$$

and

$$\mu_{V,W}(A,B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = A \oplus B, \quad \text{unit } \ 1 \in O(0)$$

Ex $B\mathbb{O}$ = bar construction on $\mathbb{O}$, B prod pres $\Rightarrow$ comm orthog monoid space. <u>Not</u> the good BO.

Ex. Additive Grassmanian $Gr(V) = \coprod_{n \geq 0} Gr_n(V)$ (finite sum for any finite V)

w/ $Gr(\alpha)(L) = \alpha(L)$

for $L \subseteq V \xrightarrow{\alpha} W$

and

$$\mu_{V,W}(L, L') = L \oplus L' \in Gr(V \oplus W)$$

Ex Multiplicative Grassmanian need more room to accept $\otimes$

$$Gr_{\otimes}(V) = \coprod_{n \geq 0} Gr_n(Sym(V))$$

unit $= \mathbb{C} \cdot 1 \in Sym(0)$

oops: $\mathbb{R} \cdot 1 \in Sym(0)$

$$Gr_{\otimes}(\alpha)(L) = Sym(\alpha)(L)$$

and $\mu_{V,W} : Gr(Sym\,V) \times Gr(Sym\,W) \longrightarrow Gr(Sym(V \oplus W))$

$$(L, L') \longmapsto L \otimes L'$$

using $Sym(V \oplus W) \underset{\text{canon}}{\cong} Sym(V) \otimes Sym(W)$

These are globally equivalent as orthogonal spaces $\rightsquigarrow$

Lemma: $i : V \longrightarrow \mathrm{Sym}(V)$ induces a global equivalence $Gr \to Gr_\otimes$

Pf: Eval @ $\mathcal{U}_K$

$$Gr(\mathcal{U}_K) = \coprod_{n\geq 0} Gr_n(\mathcal{U}_K) \xrightarrow{\cong} \coprod_{n\geq 0} Gr_n(\mathrm{Sym}(\mathcal{U}_K))$$

both are complete K-universes, hence equivalent

Not compatible w/ monoidal structures, so you get different deloopings.
[Q: Global ring space? Surely.]

Ex: $Gr = \coprod_{n\geq 0} Gr^{[n]}$

$Gr_\otimes = \coprod_{n\geq 0} Gr_\otimes^{[n]}$

$P := Gr_\otimes^{[1]}$

$P(V) = P(\mathrm{Sym}\, V)$

$\simeq B_{gl} C_2$

i.e. $\exists$ mult. model since C_2 abelian.

Generally, abelian cpt Lie gps A $\rightsquigarrow$ $B_{gl} A$ has a model as a comm. orthog monoid space.

Ex: (Not $\mathbb{Z}\times BO$) Better, BOP, periodic BO.

not significant yet, but in structure map

$$\begin{tikzcd}
BOP(V) = \coprod_{n\geq 0} Gr_n(V^2) \arrow[d] & \ni & L \leq V^2 \arrow[d, mapsto] \\
BOP(V\oplus W) = \coprod_{m\geq 0} Gr_m((V\oplus W)^2) & \ni & K_{V,W}(L\oplus(W\oplus 0))
\end{tikzcd}$$

where

$$V^2 \oplus W^2 \xrightarrow[K_{V,W}]{\cong} (V\oplus W)^2$$

$$\mu_{V,W}(L, L') = K_{V,W}(L\oplus L').$$

Then $BOP = \coprod_{m\in\mathbb{Z}} BOP^{[m]}$, $BOP^{[m]}(V) = Gr_{m+\dim V}(V^2)$

$$\mu : BOP^{[n]} \times BOP^{[m]} \longrightarrow BOP^{[n+m]}$$

$\mathbb{BO} = BOP^{[0]}$ and non-equiv. BO

and $BOP^{[n]} \underset{gl}{\simeq} BO$

so

$$Rep(1) \times BO \cong \mathbb{Z} \times BO$$

$\nwarrow$
Better

Equivariant homotopy set: Y orthog space

$$\pi_0^G Y := \operatorname*{colim}_{V \subseteq \mathcal{U}_G} \pi_0(Y(V)^G)$$

$$\underset{\text{if } Y \text{ closed}}{=} \pi_0(Y(\mathcal{U}_G)^G)$$

If $K \xrightarrow{\alpha} G$ cont. hom, get $\alpha^* : \pi_0^G(Y) \longrightarrow \pi_0^K(Y)$

(e.g. restrictions)

For closed Y,

$$\begin{tikzcd}
\pi_0^G(Y) = \pi_0(Y(\mathcal{U}_G)^G) \arrow[d] \\
\pi_0((\alpha^* Y(\mathcal{U}_G))^K) \arrow[d, "\varphi_*"] \\
\pi_0(Y(\mathcal{U}_K)^K)
\end{tikzcd}$$

$\mathcal{U}_G$ is a K-universe via α, perhaps not complete

choose (contractibly) lin. emb. $\alpha^* \mathcal{U}_G \to \mathcal{U}_K$

Facts: Contravariantly functorial ($\beta^* \circ \alpha^* = (\alpha \circ \beta)^*$), $1^* = 1$

inner automorphisms induce identity

$\underline{\pi}_0(Y) = \{\pi_0^G(Y)\} : Rep^{op} \longrightarrow Set$

Rep = cat of cpt Lie gps & conj classes of cont. homs.

$$\pi_0^G(B_{gl}K) \xleftarrow{\cong} \mathrm{Rep}(K,G) \quad \text{by} \quad \alpha^*(e_K) \longleftarrow [\alpha]$$

where

$$e_K \in \pi_0^K(B_{gl}K)$$

$$\|$$

$$[V \hookrightarrow \mathcal{U}_K]$$

i.e. $\underline{\pi}_0(B_{gl}K) = \mathrm{Rep}(-,K)$

Questions

Preview $\underline{\pi}_0 \mathbf{BOP} \cong RO$

wk product (id a.e.)

$$\prod'_{[\lambda]} O^k(\mathcal{U}_\lambda)$$

either O or Sp / $\mathbb{R}$
always U / $\mathbb{C}$.

so

aug ideal

$$\mathbb{B}①(\mathcal{U}_K) = IO(K) \times O^k(\mathcal{U}_K)$$

This is why

$$\mathbb{B}\mathbb{O} = \mathbb{B}\mathrm{OP}^{[0]} \not\simeq \mathrm{B}◎$$

(I write ① for stefan's ◎)

Model structure: can arrange level model structure where $Y \xrightarrow{f} Z$ wk equiv if $Y(V) \longrightarrow Z(V)$ is $O(V)$-weak equivalence. Then make Bousfield localization wrt essentially constant from faithful rep'n on.

Lind's cof.b's aren't the same. (proj.)

Could do flat or proj.

Another surprise. Y, Z orthog spaces give

$$\begin{tikzcd} Y \boxtimes Z(\mathcal{U}) \arrow[r] \arrow[d, phantom, "\shortparallel"] & (Y \times Z)(\mathcal{U}) \\ \operatorname*{colim}_{\substack{V \oplus W \to \mathcal{U} \\ \text{in } L}} Y(V) \times Z(W) & \end{tikzcd}$$

$\longleftarrow$ difficult to make explicit except in some special cases, e.g. $\rightarrow$

$$\begin{tikzcd} B_{gl}G \boxtimes B_{gl}K \arrow[d, equal] & \cong & B_{gl}(G\times K) \arrow[d, equal] \\ \underbrace{L(V,-)/G \boxtimes L(W,-)/K} & \cong & \underbrace{L(V\oplus W,-)/G\times K} \end{tikzcd}$$

choices here determine correct choice here

OR $\quad \underline{A} \boxtimes \underline{B} = \underline{A\times B}$

AND $\quad Y, Z$ orthog spaces, Y flat (always possible up to gl. wk equiv)

$$Y \boxtimes Z \xrightarrow{\sim} Y\times Z$$

global equivalence

$$Y(V)\times Z(W) \longrightarrow Y(U)\times Z(U)$$

$$Y(V\cong V\oplus 0 \hookrightarrow U)$$
$$\times$$
$$Z(W \cong 0\times W \hookrightarrow U)$$

Note

A comm monoid in Top w.r.t. $\times$ $\Rightarrow$ $\prod$ Eilenberg-MacLane spaces
$\Rightarrow$ $\underline{A}$ comm orthog monoid spaces

Monoids wrt $\times$ are also monoids wrt $\boxtimes$ by $\boxtimes \to \times$
but $\exists$ many more monoids wrt $\boxtimes$. And they are naturally defined.

Day's products are always closed (categorical, 1960s)