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Lecture 5 (12×45 min, 8 pages)

Last time: $H_{k-V}(S^0;\underline{\mathbb{Z}}) \cong H_k(S^V;\underline{\mathbb{Z}})$

Same methods: give $\underline{H}_{V-k}(S^0;\underline{\mathbb{Z}}) \cong H^k(S^V;\underline{\mathbb{Z}})$ "Serre dual" to homology
(∄ a dualizing class but ∃ the classes it would give you)

Thm (Dugger; Hu-Kriz)
For $G = C_2$ $\quad Gr(K_{\mathbb{R}}) = \bigvee_{n\in\mathbb{Z}} S^{n\rho}\wedge H\underline{\mathbb{Z}}$

and $Gr(MU_{\mathbb{R}}) = \bigvee_{\substack{p\in MU_* \\ p \text{ monomial}}} S^{\frac{|p|}{2}\rho}\wedge H\underline{\mathbb{Z}}$ $\quad \mathbb{Z}[x_1,x_2,x_3,\dots]$, $|x_i| = 2i$

from Schubert cell decomposition

(Milnor hypersurfaces are Real manifolds)

Get ss's $H\underline{\mathbb{Z}}_*\left(\bigvee_{n\in\mathbb{Z}} S^{n\rho}\right) \Rightarrow \pi_{\star} K_{\mathbb{R}}$

$H\underline{\mathbb{Z}}_*\left(\bigvee_{p\in L} S^{\frac{|p|}{2}\rho}\right) \Rightarrow \pi_{\star} MU_{\mathbb{R}}$

$H_{\star}(pt;\underline{\mathbb{Z}})[\bar{v}_1^{\pm 1}]$ $\quad |\bar{v}_1| = \rho$ $\qquad H_{\star}(pt;\underline{\mathbb{Z}})[\bar{x}_1,\bar{x}_2,\dots]$, $|\bar{x}_i| = i\rho$

To say what is happening in 1st s.s. w/o invoking results of Atiyah we need the 2nd.

$\Phi^{C_2} MU_{\mathbb{R}} = MO$ "Geometric fixed pts play nicely w/ Thom spectra"

$MU_{\mathbb{R}}(V) = \mathrm{Thom}\left(\begin{matrix}\xi_V\\ \downarrow \\ BU(V)\end{matrix}\right)$ $\quad$ ∴ apply Φ^{C_2} amounts to taking fixed pts. of $BU(V)$ to get $BO(V)$ & then taking Thom spectra.

But

$\Phi^{C_2} MU_{\mathbb{R}} = (MU_{\mathbb{R}}[a_\sigma^{-1}])^{C_2}$ so $\pi_*\Phi^{C_2}MU_{\mathbb{R}} = \pi_*(MU_{\mathbb{R}}[a_\sigma^{-1}])(C_2/C_2)$

$\cong (\pi_{\star} MU_{\mathbb{R}}[a_\sigma^{-1}])(C_2/C_2)$

$= \mathbb{F}_2[y_2, y_4, y_5, y_6, \ldots]$, $|y_i| = i$, $i \neq 2^j - 1$

and this is easier w/ the additional Mackey functor structure.

Φ^{C_2} commutes w/ $\vee$ and $\wedge$ so

$$\Phi^{C_2}(\mathrm{Gr}(MU_{\mathbb{R}})) = \bigvee_{p \in L} \underbrace{\Phi^{C_2}(S^{\frac{|p|}{2}\rho})}_{= S^{\frac{|p|}{2}}} \wedge \Phi^{C_2} H\underline{\mathbb{Z}}$$

Now $\tilde{E}\mathcal{P} = S^{\infty\sigma}$ w/ standard cell structure from yesterday

$\rightsquigarrow \pi_\star(\tilde{E}\mathcal{P} \wedge H\underline{\mathbb{Z}}) = \mathbb{F}_2[a_\sigma^{\pm 1}, u_{2\sigma}]$ (a_σ annih classes we didn't account for yesterday)

$\cong \mathbb{F}_2[a_\sigma^{\pm 1}, b]$, $|b| = 2$, $b = u_{2\sigma}/a_\sigma^2$

So we have a ss w/

$$E_2 = \mathbb{F}_2[a_\sigma^{\pm 1}, b, f_1, f_2, \ldots] \quad \text{where } \left(f_i : S^{\frac{|p|}{2}} \xrightarrow[a_\sigma^{\frac{|p|}{2}}]{} S^{\frac{|p|}{2}\rho}\right) \leftrightarrow \bar{x}_i$$

$|a_\sigma| = -\sigma$, $|b| = 2$, $|f_i| = i$. $\exists\, \mathbb{Z}$ graded subalg by omitting a_σ.

Have a ss w/ $E_2 = \mathbb{F}_2[b, f_1, f_2, \ldots] \Rightarrow \mathbb{F}_2[y_2, y_4, y_5, \ldots]$

Now $MU_{\mathbb{R}}$ has formal group law interpretation. Araki, Landweber, Hu-Kriz show the f_i are ∞-cycles. (Real formal groups.)

Only possibility for differentials:

$$b^{2^i} \longmapsto f_{2^{i+1}-1}$$

Can also choose generators so that $f_i \mapsto y_i$. (or def $y_i := \mathrm{Im}\, f_i$)

RO(G) graded version

$$u_{2\sigma}^{2^i} \longmapsto a_\sigma^{2^{i+2}-1}\, \bar{x}_{2^{i+1}-1}$$

Map $MU_{\mathbb{R}}$ to simpler problem (MO) where Thom told us everything. Know everything in target ss, so get info about classes in source which must support differentials.

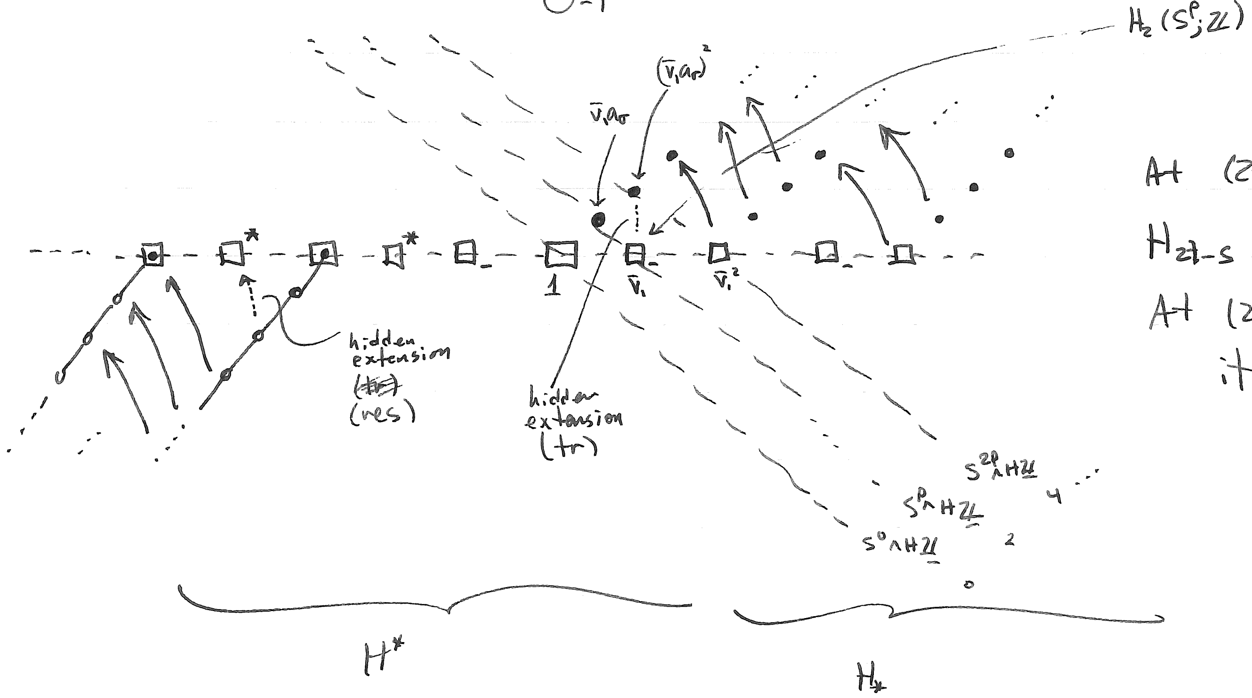
Back to K-theory. $\underline{H}_\star(pt;\underline{\mathbb{Z}})[\bar{v}_1^{\pm 1}]$

In $\mathbb{Z}$-graded stems as Mackey functors:

$\square = \underline{\mathbb{Z}}$ $\qquad$ $\square^* = \underline{\mathbb{Z}}^*$

$\square_- = \underline{\mathbb{Z}_-}$ $\qquad$ $\bullet = \begin{matrix}\mathbb{Z}/2 \\ \circlearrowright \\ 0\end{matrix}$

$\boxdot = \begin{matrix}\mathbb{Z}/2 \\ \circlearrowright \\ \mathbb{Z}_- \circlearrowleft -1\end{matrix}$



At $(2t-s, s)$ $H_{2t-s}(S^{t\rho};\underline{\mathbb{Z}})$ At $(2t-s-1, s)$ it is 0.

of regular rep'n spheres

Conner Floyd map tells the differential $\quad u_{2\sigma} \mapsto a_\sigma^3 \bar{v}_1$

so $\bar{v}_1^2 u_{2\sigma} \mapsto a_\sigma^3 \bar{v}_1^3$

$\exists$ hidden extension

$\ker(a_\sigma) = \mathrm{Im}(tr)$ forces it! $\quad a_\sigma \cdot (\bar{v}_1 a_\sigma)^2 = 0$ so $(\bar{v}_1 a_\sigma)^2 \in \mathrm{Im}\, tr$

~~Homotopy groups~~ Mackey functors

$$\begin{array}{cccccccccccc} \cdots & \square & \bullet & \boxdot & \square^* & \square_- & \square & \bullet & \boxdot & \square^* & \square_- & \cdots \\ & -8 & & -6 & -4 & -2 & 0 & 1 & 2 & 4 & 6 & \end{array}$$

$\underbrace{\hphantom{0 \quad 1 \quad 2 \quad 4 \quad 6}}_{1\text{ period}}$

Pairing : $\square^*$ in -8 times $\square$ in $+8$ gives 2 in π_0, not 1

The hidden extension gives an element in -8 whose product with the class in 8 gives 1.

Similarly for the other hidden extension.

Periodicity doesn't show up at E_∞.

— BREAK —

Two things left to do. Use $MU_{\mathbb{R}}$ & see how it implies classical information.

Hopefully:

$$S^0 \xrightarrow{\eta} MU \underset{\eta_R}{\overset{\eta_L}{\rightrightarrows}} MU \wedge MU \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; MU \wedge MU \wedge MU \quad \cdots$$

Comm ring spectrum gives ~~&~~ maps back to produce a cosimplicial comm ring

$$MU^{\wedge \bullet}$$

BK ss $\rightsquigarrow$ cocham cx $\rightsquigarrow$ gr ab gp $\rightsquigarrow$

like $\quad k \longrightarrow k[G] \rightrightarrows k[G\times G] \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; k[G\times G\times G] \quad \cdots$

Powerful tool for $\pi_* S$ since $\pi_* MU = L = \mathbb{Z}[x_1, x_2, \ldots]$

$$\pi_*(MU \wedge MU) = W \cong MU_*[b_1, b_2, \ldots] \qquad |b_i| = 2i$$

one map is constant poly; other is a mess: better to say

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what it corepresents: $L: R \mapsto$ formal group laws on R } a category
$W: R \mapsto$ strict iso's of fgl/R

The two maps are source & target for the iso
the mult gives
& then there is composition.

$$\pi_*(MU^{\wedge 3}) \cong W \underset{L}{\otimes} W$$

3 fgl's & iso's
one iso, the other, or their compsite

$$E_2^{s,t} = \operatorname{Ext}^{s,t}_{\substack{MU_* MU \\ \text{comodules}}}(MU_*, MU_*) \Rightarrow \pi_{t-s} S^0$$

$\pi_*(MU^{\wedge \bullet})$ = Cobar so this is clear.

Do this equivariantly with $MU_{\mathbb{R}}$. We lose control.

$\pi_* MU_{\mathbb{R}}$ is a mess. But $\exists$ nice filtration from

$Gr(MU_{\mathbb{R}}) = \bigvee_p S^{\frac{|p|}{2}\rho} \wedge H\underline{\mathbb{Z}}$. Now $MU_{\mathbb{R}}$ is flat, so $MU_{\mathbb{R}} \wedge MU_{\mathbb{R}}$

$$Gr(MU_{\mathbb{R}} \wedge MU_{\mathbb{R}}) \simeq \bigvee_{r \in \mathbb{Z}[b_i]} S^{\frac{|r|}{2}\rho} \wedge MU_{\mathbb{R}}$$

(pick off coeffs in iso of fgl's
Hu & Kriz after Araki)

Recall UCT & Box Tor ugliness: integer graded homotopy here is not ~~flat~~ integer graded homotopy of $MU_{\mathbb{R}}$. If we use $RO(G)$ graded modules, it is flat. So hom alg is much simpler. Again $Gr(MU^{\wedge 2}) \simeq \bigvee_{W} S^{\frac{|r|}{2}\rho} \wedge H\underline{\mathbb{Z}}$

$$Gr(MU^{\wedge 3}) \simeq \bigvee_{W \otimes_{L} W} S^{\frac{|r|}{2}\rho} \wedge H\underline{\mathbb{Z}}, \text{ etc.}$$

Get $E_2 = H_\star(pt; \underline{\mathbb{Z}}) \otimes Ext^{s, \frac{t}{2}\rho}_{\substack{MU_* MU \\ \text{comod}}}(MU_*, MU_*)$

slice style diff's

$$\Downarrow$$

$$Ext^{\star\star}_{\substack{MU_{\mathbb{R}\star} MU_{\mathbb{R}} \\ \text{Comod}}}(MU_{\mathbb{R}\star}, MU_{\mathbb{R}\star})$$

Adams style

$$\Downarrow$$

$$\pi_{\star - s} S^0$$

<u>OR</u>: Do a double complex & go directly to $\pi_{\star - s} S^0$.

Motivic version gives same sort of spectral sequences.

2 replaced by ρ

& diff's preserve wt $= \#$ sign rep'ns.

<u>AND</u>

Take Φ^G: works well w/ all of this

get MO's, a wedge of $H\mathbb{F}_2$'s. So ordinary Adams res. by stripping out all copies of sign rep'n.

$$\mathbb{F}_2[b] \otimes Ext^{s, t/2}_{A}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_* S^0 \leftarrow \text{2-complete, of course.}$$

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Adams Novikov $\xrightarrow{\text{take fixed pts}}$ ordinary Adams

(bizarre?)

Norms A simple way to think about norms

Symm Monoidal Category

$\rightsquigarrow$ [In the problem session]

Ind & Coind both built by

- take a bunch of copies of an H object
- Do a G/H indexed Symm Mon Product of copies of H-object

$\wedge$ = ~~coprod~~ product in Comm Ring

so ~~res~~

ind = left adjoint to forget

so norm = — " —

So $N_H^G i_H^* R \longrightarrow R$ ring map

$i_H^* \tilde{E}P \simeq *$ so

$$
\begin{tikzcd}
 & S^0 \arrow[dl] \arrow[dr] & \\
\underbrace{N_H^G i_H^* \tilde{E}P}_{\simeq *} \wedge S^0 \arrow[rr, "\times"] & & \tilde{E}P \wedge S^0
\end{tikzcd}
$$

$\not\exists!$

N_H^G give norms, Tambara functor, hard to remember their relations. No, here's the easy way

Distrib Law

Q: what happens in Burnside ring?

There $N_H^G T$ is actually coind $F_H(G, T)$

so Norm of a transfer is a generalized transfer.

Binom Coeffs $\leftrightarrow$ ways to group things
$\leftrightarrow$ subsets
$\leftrightarrow$ functions to $\{0,1\}$