Mike Hill
GHaHT
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Lecture 4

(2×45 min, 9 pages)

Back to computing.

$$\begin{tikzcd}
C_2 \times C_2 \arrow[d, "\gamma \times 1"'] & \cong & C_2 \arrow[drr, "\gamma"] & \amalg & C_2 \arrow[dll, "\gamma"'] \\
C_2 \times C_2 & \cong & C_2 & \amalg & C_2
\end{tikzcd}$$

can be done using action on either factor & the two are related by a twist

Check: $\left\{ \begin{matrix} (e,e) \\ (\gamma,\gamma) \end{matrix} \right\}, \left\{ \begin{matrix} (e,\gamma) \\ (\gamma,e) \end{matrix} \right\}$

$$\begin{tikzcd}
 & \underline{m}_{C_2}(C_2) \arrow[d, equal] & \underline{m}_{C_2}(C_2 \arrow[d, equal] \\
\text{so} & \underline{M}(C_2 \times C_2) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & \underline{M}(C_2 \times C_2) \arrow[d, phantom, sloped, "\cong"] \\
 & \underline{M}(C_2) \oplus \underline{M}(C_2) \arrow[r, "\begin{bmatrix} 0 & \gamma \\ \gamma & 0 \end{bmatrix}"'] & \underline{M}(C_2) \oplus \underline{M}(C_2)
\end{tikzcd}$$

Prototype for such.

Today: Compute Bredon (co)homology of repn spheres S^V for C_{p^n}.

Irreps: p even: $1, \overset{\text{sign}}{\sigma}, \lambda(k) : C_{2^n} = \mu_{2^n} \hookrightarrow S^1$ $\quad 1 \leq k \leq 2^{n-1} - 1$

$$\begin{tikzcd}
\mu_{2^n} \arrow[r, hook] \arrow[dr] & S^1 \arrow[d, "\cdot k"] \\
 & S^1
\end{tikzcd}$$

p odd: $1, \lambda(k) : \mu_{p^n} \longrightarrow S^1$

$$\begin{tikzcd}
\mu_{p^n} \arrow[r] \arrow[dr] & S^1 \arrow[d, "\cdot k"] \\
 & S^1
\end{tikzcd}$$

S^V cares only about $\nu_p(k)$ $\quad$ p-locally (so $\underbrace{\text{p-localized}}_{\text{whatever algebraically localizes all the Mackey functor gps.}}$ @ each f.p.)

So for C_4: $1, \sigma, \lambda$

Bredon homology of trivial spheres already done.

σ factors through C_2 so CW structure is the same: $S^0 \cup (C_4/C_{2+} \wedge e^1) = S^\sigma$

(Do this generally: take a faithful repn of a quotient & pull it back.)

$\underline{m} = \underline{\mathbb{Z}}$:

$$\begin{tikzcd}
\mathbb{Z} \arrow[d, bend right] & \mathbb{Z} \arrow[l, "2"'] \arrow[d, bend right, "\Delta"'] \\
\mathbb{Z} \arrow[u, bend right, "2"'] \arrow[d, bend right] & \mathbb{Z}^2 \arrow[l, "\nabla"'] \arrow[u, bend right, "\nabla"'] \arrow[d, bend right] \\
\mathbb{Z} \arrow[u, bend right, "2"'] & \mathbb{Z}^2 \arrow[l, "\nabla"'] \arrow[u, bend right, "2"']
\end{tikzcd}$$

middle map determines the rest by transfer and restriction. It is the fold ∇. Then bottom $= \nabla$ and top $= 2$

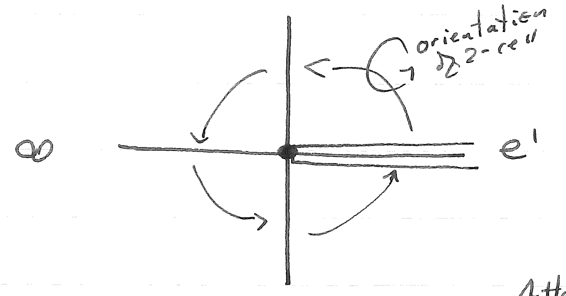
No label $\Leftrightarrow$ Id or obvious (e.g. 0)

Now, ker & coker

$$\begin{tikzcd}
\underline{\text{ker}} & \underline{\text{cok}} \\
\mathbb{Z}/2 \arrow[d, bend right] & 0 \\
0 \arrow[u, bend right] \arrow[d, bend right] & \mathbb{Z}_- \arrow[d, bend right] \\
0 \arrow[u, bend right] & \mathbb{Z}_- \arrow[u, bend right, "2"']
\end{tikzcd}$$

Sanity check: Drop top row, get the C_2 computation. That's the S^1 with ~~sign~~ conjugation action.

S^λ:



2-skeleton = 4 quadrants

Attach:

$$\begin{tikzcd}
S^0 \cup (C^4_+ \wedge e^1) & \cup\, (C^4_+ \wedge e^2) \\
\pi_e^{C_4} \arrow[u] & (C_{4+} \xrightarrow{1-\gamma} C_{4+}) \wedge S^1 \arrow[u]
\end{tikzcd}$$

Complex

$$\begin{tikzcd}[row sep=large, column sep=huge]
\mathbb{Z} \arrow[d, bend right] & \mathbb{Z} \arrow[l, "4"'] \arrow[d, bend right, "\Delta"'] & \mathbb{Z} \arrow[l, "0"'] \arrow[d, bend right, "\Delta"'] \\
\mathbb{Z} \arrow[u, bend right, "2"'] \arrow[d, bend right] & \mathbb{Z}^2 \arrow[l, "2\nabla"'] \arrow[u, bend right, "\nabla"'] \arrow[d, bend right, "\Delta"'] & \mathbb{Z}^2 \arrow[l, "1-\gamma"] \arrow[u, bend right, "\nabla"'] \arrow[d, bend right, "\Delta"'] \\
\mathbb{Z} \arrow[u, bend right, "2"'] & \mathbb{Z}^4 \arrow[l, "\nabla"] \arrow[u, bend right, "\nabla"'] & \mathbb{Z}^4 \arrow[l, "1-\gamma"] \arrow[u, bend right, "\nabla"']
\end{tikzcd}$$

start with underlying

$$\begin{tikzcd}
2 \arrow[d, mapsto] & \begin{pmatrix}1\\0\end{pmatrix} \arrow[l, dashed, "2\nabla"'] \arrow[d, mapsto] \\
2 & \begin{pmatrix}1\\0\\1\\0\end{pmatrix} \arrow[l, mapsto]
\end{tikzcd}$$

Homology

$$\begin{tikzcd}[column sep=huge]
\mathbb{Z}/4 \arrow[d, bend right, two heads] & & \mathbb{Z} \arrow[d, bend right] \\
\mathbb{Z}/2 \arrow[u, bend right, hook] & 0 & \mathbb{Z} \arrow[u, bend right, "2"'] \arrow[d, bend right] \\
0 & & \mathbb{Z} \arrow[u, bend right, "2"']
\end{tikzcd}$$

$$0 \qquad \underline{\mathbb{Z}}$$

$H_2(S^\lambda; \underline{\mathbb{Z}}) = \underline{\mathbb{Z}}$

Def: V is $\underline{\mathbb{Z}}$ orientable if

$$H_{\dim V}(S^V; \underline{\mathbb{Z}}) = \underline{\mathbb{Z}}.$$ →

Orientation class $u_V \in H_{\dim V}(S^V; \underline{\mathbb{Z}})(C/C)$ is a choice of generator.

Note: S^σ is not C_4 ~~orientability~~ orientable, but restricted to C_2 it becomes orientable. $\left(\underline{\mathbb{Z}}_-\big|^{C_4}_{C_2} = \underline{\mathbb{Z}}\right)$

$$V \text{ is } \underline{\mathbb{Z}}\text{-orientable} \iff \begin{tikzcd} G \arrow[r] \arrow[dr, dashed] & O(V) \\ & SO(V) \arrow[u, hook] \end{tikzcd}$$

So p odd $\Rightarrow$ all rep'ns are orientable, and
$p = 2 \Rightarrow$ (V orientable $\iff$ (it contains an even number of sign representations))

$$\Rightarrow \quad u_{2k\sigma} \in H_{2k}(S^{2k\sigma}; \underline{\mathbb{Z}})$$
$$u_{\lambda(k)} \in H_2(S^{\lambda(k)}; \underline{\mathbb{Z}}).$$

Note: We'd be better off writing Mackey functor (w/o Weyl action) as $\mathbb{Z}[C_4]$ representations

$(\underline{\mathbb{Z}})_{C_{p^n}/C_{p^k}}$:

C_{p^n}/C_{p^n}	$\mathbb{Z} = (\ \cdots\ )^{C_{p^n}}$	
$\vdots$	$\vdots$	$\vdots$
$C_{p^n}/C_{p^{k+1}}$	$\left(\mathbb{Z}[C_{p^n}] \otimes_{\mathbb{Z}[C_{p^k}]} \mathbb{Z}\right)^{C_{p^{k+1}}}$	
C_{p^n}/C_{p^k}	$\mathbb{Z}[C_{p^n}] \otimes_{\mathbb{Z}[C_{p^k}]} \mathbb{Z}$	$\mathbb{Z}[C_{p^n}] \otimes_{\mathbb{Z}[C_{p^k}]} \underline{M}\vert C_{p^n}/C_{p^k})$
$\vdots$	$\vdots$	$\vdots$
C_{p^n}/e	$\mathbb{Z}[C_{p^n}] \otimes_{\mathbb{Z}[C_{p^k}]} \mathbb{Z}$	$\mathbb{Z}[C_{p^n}] \otimes_{\mathbb{Z}[C_{p^k}]} M(C_{p^n}/e)$

— Break —

Q: $RO(G)$ graded homology of a point w/ $\underline{\mathbb{Z}}$ coeff's for almost any G? Little is known.
or coeffs in Burnside or representation coeffs.

Easy case: if the coeffs are a Mackey field,
e.g. Galois fixed pts of a field extension.

Images of transfers always give an ideal.
Mackey fields behave like fields.
Are they equipped w/ Norm maps?
Not if pulled back along a quotient that is nontrivial.

— END OF BREAK —

$S^0 \xrightarrow{a_V} S^V$ gives $(S^0 \longrightarrow S^V \wedge H\underline{\mathbb{Z}}) \in H_0(S^V; \underline{\mathbb{Z}})(G/G)$, we'll also call a_V.

<u>Prop</u> (i) $\mathrm{res}^G_H a_V = a_{\mathrm{res}^G_H V}$
(ii) If $V^H \neq \{0\}$ then $[G/H] \cdot a_V = tr^G_H \circ \mathrm{res}^G_H a_V = 0$
Hence $[G:H]\, a_V = 0$ in $H_0(S^V; \underline{\mathbb{Z}})(G/G)$.

Torsion in homology but rationally nontrivial in $\pi_\star$.

$S^0 \xrightarrow{a_\sigma} S^\sigma \xrightarrow{a_\sigma} S^{2\sigma}$

Chain cx for

$$\begin{tikzcd}
S^{2\sigma}: & \underline{\mathbb{Z}}_1 & \underline{\mathbb{Z}}_{C_2} \arrow[l, "\nabla"'] & \underline{\mathbb{Z}}_{C_2} \arrow[l, "1-\gamma"'] \ni u_{2\sigma} \text{ gen} \\
S^{\sigma}: \arrow[u, hook] & \underline{\mathbb{Z}}_1 \arrow[u, "\cdot a_\sigma"'] & \underline{\mathbb{Z}}_{C_2} \arrow[l, "\nabla"'] \arrow[u] & \\
S^0 \arrow[u, hook] & \underline{\mathbb{Z}}_1 \arrow[u, "\cdot a_\sigma"'] \ni \underline{1} = u_0 & &
\end{tikzcd}$$

iso on
ch cx
level

so in $C_0(S^{2\sigma})$ gen is a_σ^2

$$\begin{tikzcd}
S^0 \arrow[d] & & & & \\
S^\sigma \arrow[d] & & & & \\
S^{2\sigma} \arrow[d] & \underline{\mathbb{Z}}_1 \arrow[d, "a_\sigma"] & \underline{\mathbb{Z}}_{C_2} \arrow[l] \arrow[d, "a_\sigma"] & \underline{\mathbb{Z}}_{C_2} \arrow[l] \arrow[d, "a_\sigma"] & \\
S^{3\sigma} & \underline{\mathbb{Z}}_1 & \underline{\mathbb{Z}}_{C_2} \arrow[l, "\nabla"'] & \underline{\mathbb{Z}}_{C_2} \arrow[l, "1-\gamma"'] & \underline{\mathbb{Z}}_{C_2} \arrow[l, "1+\gamma"'] \\
 & a_\sigma^3 \in & \cancel{a_\sigma^2 u_\sigma} & a_\sigma u_{2\sigma} & \cancel{u_{3\sigma}} \text{ not orientable} \arrow[ll, bend left]
\end{tikzcd}$$

Rmk: $a_{V\oplus W} = a_V a_W$ and $u_{V\oplus W} = u_V u_W$

actual, not virtual, rep'ns.

Thm: For $\star \in \mathbb{Z}\text{-Rep}(C_4) \subseteq RO(C_4)$,

$$\pi_\star(H\underline{\mathbb{Z}}) = \mathbb{Z}[a_\sigma, a_\lambda, u_{2\sigma}, u_\lambda] \Big/ \begin{array}{l} 2a_\sigma = 4a_\lambda = 0, \\ a_\sigma^2 u_\lambda = 2a_\lambda u_{2\sigma} \end{array}$$

degrees $-\sigma,\ -\lambda,\ 2-2\sigma,\ 2-\lambda$

if $\star = k - V$ then

$$\pi_{k-V} = \pi_k(S^V \wedge H\underline{\mathbb{Z}}) = H_k(S^V; \underline{\mathbb{Z}}_1)(C_4/C_4)$$

Ex: $S^{2\sigma+3\lambda}$: two choices to get -2σ : $\left\{ \begin{array}{c} u_{2\sigma}, \\ a_\sigma^2 \end{array} \right\} \cdot \left\{ \begin{array}{c} a_\lambda^3 \\ a_\lambda^2 u_\lambda \\ a_\lambda u_\lambda^2 \\ u_\lambda^3 \end{array} \right\}$ to get -3λ

but
$$\begin{aligned} a_\sigma^2 \cdot u_\lambda^3 &= (a_\sigma^2 u_\lambda) u_\lambda^2 \\ &= 2a_\lambda u_{2\sigma} u_\lambda^2 \end{aligned}$$

Each a. is p-power torsion and tells how far down you can restrict before becoming 0.

For Q_{2^n} brackets involving relations above generate add'l classes.

S^V for $C_{p^n} \supseteq C_{p^{n-1}} \supseteq \cdots$

$V^{C_{p^n}} = a_0 \cdot 1$, etc, so $V = \sum_{\substack{m=1\\ 0?}}^{n} a_{n-m} \cdot \lambda(p^m)$

$\lambda(p^n) = 1$
($\mathbb{C}$, so $\mathbb{R}^2$)

If $v \in \lambda(p^k)$ then C_{p^k} fixes v.

$$\begin{matrix} V^{C_{p^n}} & \subseteq & V^{C_{p^{n-1}}} & \subseteq & \cdots \\ \| & & \| & & \\ a_0 \cdot 1 & & a_0 \cdot 1 + a_1 \lambda(p^{n-1}) & & \cdots \end{matrix}$$

gives a cell structure for S^V such that

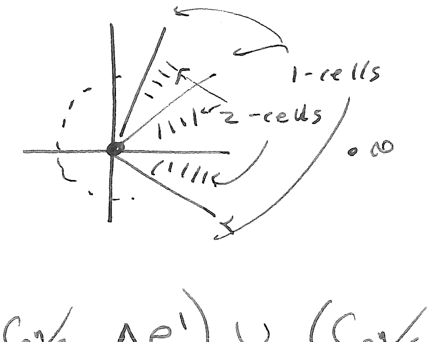
- $\overset{\text{even/odd}}{\vee}$ skeleta are all sub representation spheres (parity depending on a_0)

Explicitly: $S^{a_0} \subseteq S^{a_0 + \lambda(p^{n-1})} \subseteq S^{a_0 + 2\lambda(p^{n-1})} \subseteq \cdots \subseteq S^{a_0 + a_1\lambda(p^{n-1})} \subseteq$

$S^{a_0 + a_1\lambda(p^{n-1}) + \lambda(p^{n-2})} \subseteq \cdots \subseteq S^{a_0 + a_1\lambda(p^{n-1}) + a_2\lambda(p^{n-2})} \subseteq$

$\cdots \subseteq S^V$.

Proof: $\lambda(p^k)$: $C_{p^n} \longrightarrow C_{p^n}/C_{p^k} \hookrightarrow S^1$



$$S^{\lambda(p^k)} = S^0 \underset{\pi^{C_{p^n}}_{C_{p^k}}}{\cup} (C_{p^n}/C_{p^k} \wedge e^1) \underset{1-\gamma}{\cup} (C_{p^n}/C_{p^k} \wedge e^2)$$

Then use:

Let V_1 be s.t. $i^*_{C_{p^k}} V_1 \cong \mathbb{R}^{\dim V_1}$ (trivial rep'n)

$$\text{Then } S^{V_1}\wedge S^{\lambda(p^k)} = S^{V_1} \cup \underbrace{\left(S^{V_1}\wedge C_{p^n}/C_{p^k}\wedge e^1\right)}_{\cong\, C_{p^n+}\wedge_{C_{p^k}} e^{\dim V_1+1}} \cup \underbrace{\left(S^{V_1}\wedge C_{p^n}/C_{p^k}\wedge e^2\right)}_{C_{p^n+}\wedge_{C_{p^k}} e^{\dim V_1+2}}$$

$$\| \qquad S^{V_1\oplus\lambda(p^k)}$$

$$\begin{tikzcd} \pi^{C_{p^n}}_{C_{p^k}} & 1-\gamma \\ \nabla,\ \text{fold} \arrow[u, squiggly] & 1-\gamma \arrow[u, squiggly] \end{tikzcd}$$

p odd

(a sign shows up) V_1 non-orientable $1+\gamma$

$$e^{\dim V_1+1} \cong i^*_{C_{p^k}} S^{V_1}\wedge e^1$$

This trick won't handle elem abel or quaternion.

Note: sequence of inclusions of skeleta consists of multiplications by $a_p, a_q, \dots, a_1, a_\lambda, a_\lambda, \dots, a_\lambda, a_{\lambda(p^2)}, \dots$

Key: Singular set is always a sphere.

This also tells how to build a chain complex for $S^{V+\lambda(p^k)}$ where $i_{C_{p^k}} V = \mathbb{R}^{\dim V}$. Then

$$\underline{C}_*\left(S^{V+\lambda(p^k)}\right) = \begin{cases} \underline{C}_*(S^V) & * \le \dim V \\ \underline{\mathbb{Z}}_{C_{p^n}/C_{p^k}} & \dim V < * \le \dim V+2 \\ 0 & \text{else} \end{cases}$$

Last two attaching maps are induced, so determined by value at C_{p^n}/C_{p^k}, $i^*_{C_{p^k}} S^{V+\lambda(p^k)} = S^{\dim V+2}$.

Every sub rep'n sphere showed up in 2 flavors:
homology of subsphere times $a^{*}_{\lambda(p^k)}$

If everything is orientable (e.g p odd)

$$H_*(S^{V+\lambda(p^k)};\underline{\mathbb{Z}}) = \begin{cases} a_{\lambda(p^k)} H_*(S^V) & * \le \dim V \\ u_{V+\lambda(p^k)} & * = \dim V + 2 \\ 0 & \text{else} \end{cases}$$

and in nonorientable case, a_σ's show up.

∃ Kunneth ss from derived functors of $\boxtimes$ to π_* (smash)
Everything has ∞ hom. dim.
Easier to tensor chain complexes, since box product of permutation modules is just perm. on Cartesian product.

Main problem: nice CW structure. For perm. rep'ns ought to be able to use simplicial model.

Also good model for regular rep'n.

Next: $C_2\times C_2$ (since Q_8 and D_8 surject, need this first to get them).

Borel-Smith condition tells you the dims of the trivial parts.

Box w/ Burnside ring okay. ($\boxtimes$ unit)
Others in image of induction okay since induction is exact.

Proj $\not\Rightarrow$ flat according to Lewis.

What is $\boxtimes$-Tor? (See Lewis' paper.)

Flat resolutions not unique up to chain homotopy, so meaning of calculation unclear.

Lewis' examples are cpt Lie, not finite. (See MathReview)

Ind & CoInd not available.