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Lecture 3 (2×45 min, 6 pages)

Homology: given a Mackey functor $\underline{M}$ we get two functors

$$sMan^{G\,op} \longrightarrow dgAb$$

via

$$sMan^{G\,op} \xrightarrow{\underline{M}} sAb \longrightarrow dgAb$$

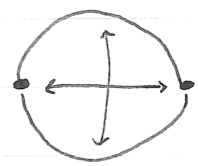
using restrictions get a cochain complex
transfers ——— " chain complex.

Def: $H_*(X_\cdot; \underline{M}) = H_*(\underline{M}_*(X_\cdot))$ for $X_\cdot \in sMan^G$

$H^*(X_\cdot; \underline{M}) = H^*(\underline{M}^*(X_\cdot))$

"Bredon homology & cohomology"

Ex: $0 \quad C_2 = \{e, \gamma\} \quad d_0(e) = e$

$1 \quad C_2 \rightrightarrows \quad d_1(e) = \gamma$ has Geom realization  $= S(2\sigma)$

$\underline{M} \mapsto$

$$\begin{array}{c} \underline{M}(C_2/e) \\ \big| \; d_0 - d_1 = 1-\gamma \\ \underline{M}(C_2/e) \end{array}$$

γ acting by Weyl group action

Easy to calc H_* and H^*: the Mackey functor tells all. Just as H_*, H^* of a simplicial complex is easy.

Ex: $0 \quad * \amalg *$

$1 \quad * \quad C_2$

Attach a C_{2+} 1-cell to the 0-sphere $C_{2+} \to S^0 \to S^\sigma$

reduced $\underline{M} \mapsto$

$$\begin{array}{c} \underline{M}(C_2/C_2) \\ \big| \; \underline{M}(\pi_e^{C_2}) \\ \underline{M}(C_2/e) \end{array}$$

$H_* =$ homology of

$$\begin{array}{cl} \underline{M}(C_2/C_2) & 0: \operatorname{cok}(tr) \\ \uparrow tr_e^{C_2} & \\ \underline{M}(C_2/e) & 1: \ker(tr) \end{array}$$

$$H^* \left(\begin{array}{c} \underline{M}(C_2/C_2) \\ \downarrow res_e^{C_2} \\ \underline{M}(C_2/e) \end{array} \right)$$

similarly

For $\underline{M} = \underline{\mathbb{Z}}$

$res = Id$

$tr = 2$

$$H_*(S^\sigma; \underline{\mathbb{Z}}) = \begin{cases} 0 & * \neq 0 \\ \mathbb{Z}/2 & * = 0 \end{cases}$$

$$H^*(S^\sigma; \underline{\mathbb{Z}}) = 0$$

This forgets structure. Can take $\underline{M} : \mathcal{S}et^G \longrightarrow \mathcal{M}ackey^G$

$T \longmapsto \underline{M}_T$ as defined yesterday.

Co & Contra-variant forms.

$\underline{M}_{G/H}$ has a map of Mackey functors to $\underline{M} = \underline{M}_{G/G}$. (covariant)

and

$\underline{M} \longrightarrow \underline{M}_{G/H}$ (contra)

Do computations from before, producing Mackey functors, not just abel. gps.

$$\begin{tikzcd}
\underline{M}_{C_2} \arrow[d, "1-\gamma"] \\
\underline{M}_{C_2}
\end{tikzcd}$$

$tr(a, -\gamma a) = a - \gamma(a) \longleftarrow\!\!| \quad a = tr(a, 0)$

$\underline{M}_{C_2}$ is

$$\begin{tikzcd}
\underline{M}(C_2/e) \arrow[d, bend right, "\Delta"'] & & \underline{M}(C_2/e) \arrow[d, bend right, "\Delta"'] \\
\underline{M}(C_2/e) \oplus \underline{M}(C_2/e) \arrow[u, bend right, "\nabla"'] \arrow[loop below, "\text{Weyl}"] & & \underline{M}(C_2/e) \oplus \underline{M}(C_2/e) \arrow[u, bend right, "\nabla"'] \arrow[ll, "1-\tau"] \arrow[loop below]
\end{tikzcd}$$

forced by $1-\gamma$, and this completely determines the map.

~~BUT, Not 0, since it is really $a - \gamma(a)$~~

f.p. level

Another example: $\underline{M}_{C_2}$ is

$$\begin{tikzcd}
\underline{M}_{C_2} \arrow[d, dash] & m(C_2/C_2) \arrow[d, bend right, "res"'] \arrow[r] & M(C_2/e) \arrow[d, bend right, "\Delta"'] \\
\underline{M} & m(C_2/e) \arrow[u, bend right, "tr"'] \arrow[r, "\binom{1}{\gamma}"'] & m(C_2/e) \oplus m(C_2/e) \arrow[u, bend right, "\nabla"']
\end{tikzcd}$$

underlying level

$\leftarrow$ Weyl invariant diagonal

for cohomology

Now res in target inj gives

$b \longmapsto$ hence $res\, b$

$res(b) \longmapsto (res\, b, \gamma\, res\, b) = (res\, b, res\, b)$

since $res \subset$ Weyl inv.

[follows from double coset formula: $C_2 \times C_2 = C_2 \amalg C_2$

Write out this calculation to understand it]

(i.e. why we get twisted diagonal.)

$C_2 \times C_2$ has 2 C_2 actions: one gives $\oplus$ decomposition, the other then looks like the swap map.

Homology version

$$\begin{tikzcd}
\mathrm{tr}_e^{C_2}(a) & & a=\mathrm{tr}(a,0) \arrow[ll, dashed, leftarrow] \\
M(C_2/C_2) \arrow[d, bend right] & & M(C_2/e) \arrow[d, bend right] \\
M(C_2/e) \arrow[u, bend right] & & M(C_2/e)^{\oplus 2} \arrow[u, bend right, "\mathrm{tr}"'] \\
a+\gamma b & & (a,b) \arrow[ll, mapsto] \\
a \arrow[uuuu, mapsto, bend left] & & (a,0) \arrow[ll, leftrightarrow] \arrow[uuuu, mapsto, bend left]
\end{tikzcd}$$

At f.p. covariant version sees tr
contra ——— " ——— res.

<u>Cok</u>

$$\begin{tikzcd}
\mathrm{cok}(\mathrm{tr}) \arrow[d, bend right] \\
0 \arrow[u, bend right]
\end{tikzcd}$$

<u>Ker</u>

$$\begin{tikzcd}
\ker(\mathrm{tr}) \arrow[d, bend right] \\
\{(a,-\gamma a)\} \arrow[u, bend right]
\end{tikzcd}$$

"twisted anti diagonal"
$M(C_2/e)\otimes\sigma$

<u>$\mathbb{Z}$ case</u>

	0	1		0	1
H_*:	$\begin{tikzcd} \mathbb{Z}/2 \arrow[d, bend right] \\ 0 \arrow[u, bend right] \end{tikzcd}$	$\begin{tikzcd} 0 \arrow[d, bend right] \\ \mathbb{Z}(-) \arrow[u, bend right] \end{tikzcd}$ = sign rep'n	H^*	$\begin{tikzcd} 0 \arrow[d, bend right] \\ 0 \arrow[u, bend right] \end{tikzcd}$	Exercise (0) $\begin{tikzcd} \arrow[d, bend right] \\ \mathbb{Z} \arrow[u, bend right] \end{tikzcd}$

$\exists$ res $s\mathcal{S}et^G \longrightarrow s\mathcal{S}et^H$ & computing there is same as looking at value of Mackey functor at H:

$H_*(S^1)$: $\quad 0 \quad \mathbb{Z}(-)$
now gp acts

$H^*(S^1)$: $\quad 0 \quad \mathbb{Z}(-)$

Def: If X is a G-space $\underline{H}_*(X; \underline{M}) = H_*(\mathrm{Sing}(X); \underline{M})$
$H^*(X; \underline{M}) = H^*(\mathrm{Sing}(X); \underline{M})$ } normally called Bredon homology or cohomology of spaces

Usual properties from 1st yr alg top.
Better $\underline{H}_*$ and $\underline{H}^*$: Mackey functor valued.

Exer: $\underline{H}_*(G/H; \underline{M})$ and $\underline{H}^*(G/H; \underline{M})$
$\mathrm{Sing}_n(T) = T$ for finite G set T and d_i are all 1 so $\partial = 0$ or 1 as in non-eq. So

$$\underline{H}_*(T; \underline{M}) = \underline{M}_T$$

Similarly

$$\tilde{\underline{H}}_*(T \wedge S^n; \underline{M}) = \begin{cases} \underline{M}_T & * = n \\ 0 & * \neq n \end{cases}$$

So there ought to be a cellular theory since spheres have homology concentrated in a single degree. Cellular homology

$$\underbrace{C_n^{cell}(X) = \underline{M}(T_n)}_{Ab} \quad \text{or} \quad \underbrace{\underline{M}_{T_n}}_{Mackey} \qquad T_n = \text{set of } n\text{-cells}$$

Differentials are G-equivariant degrees.

— BREAK —

More on cellular & some Mackey functors.

$\mathcal{A}$: obj = finite G-sets
mor = Group completion $\left(\left\{ \left[S \leftarrow X \rightarrow T \right] \right\}, \amalg \text{ in } X \text{ coordinate} \right)$
$\mathcal{A}(S,T)$

iso classes to get a category or look at higher category of actual G-sets.

prod = coprod = $\amalg$
$\mathcal{A} \cong \mathcal{A}^{op}$ (Easiest pf that $\Pi = \amalg$)

Mackey functor = additive functor $\mathcal{A} \longrightarrow Ab$

Composition

$$\begin{tikzcd}
& & X \times_T Y \arrow[dl] \arrow[dr] \arrow[ddll, bend right] \arrow[ddrr, bend left] & & \\
& X \arrow[dl] \arrow[dr] & & Y \arrow[dl] \arrow[dr] & \\
S & & T & & U
\end{tikzcd}$$

res $\quad m^*\left(\begin{smallmatrix} X \\ \downarrow \\ S \end{smallmatrix}\right)$ $\qquad$ tr $\quad m_*\left(\begin{smallmatrix} X \\ \downarrow \\ T \end{smallmatrix}\right)$

Translation to usual description: $M\left(S \leftarrow X \rightarrow T\right) = m(S) \longrightarrow m(X) \longrightarrow m(T)$

$\exists$ fully faithful $\quad \mathcal{A} \hookrightarrow SW =$ Spanier Whitehead cat $\left\{ \begin{array}{l} \text{finite } G\text{-CW complexes} \\ \text{w/ stable homotopy} \\ \text{classes of } G\text{-maps} \\ \text{using a complete} \\ G\text{-universe} \end{array} \right\}$

$T \longmapsto T_+$

RO(G) grading.

Prop: For any Mackey functor $\underline{M}$ $\exists$ a spectrum $H\underline{M}$ s.t. $\underline{\pi}_* H\underline{M} = \begin{cases} \underline{M} & * = 0 \\ 0 & \text{other} \end{cases}$

Sketch:

$\underline{\pi}_k(S^n) = 0$ if $k < n$ and $\underline{A}$ if $k = n$.

$\Rightarrow$

$$\underline{\pi}_k(S^n \wedge G/H_+) = \begin{cases} 0 & k < n \\ \underline{A}_{G/H} & k = n \end{cases}$$

① So build something so that $\underline{\pi}_{<0} X = 0$ and $\underline{\pi}_0 X = \underline{M}$. Choose a proj res of $\underline{M}$: $0 \leftarrow \underline{M} \leftarrow \underline{P}_0 \leftarrow \underline{P}_1 \leftarrow \cdots$ w/ each $\underline{P}_i$ a direct sum of $\underline{A}_{G/H}$'s, $\bigoplus_{J_i} \underline{A}_{G/H}$

$$\bigvee_{J_1} G/H_+ \longrightarrow \bigvee_{J_0} G/H_+ \longrightarrow X$$

② Kill all maps from higher spheres.

Functorial in $\underline{M}$ in the homotopy category since proj res is. //

[Same strategy for global stable homotopy theory.]

Now: $H^*(X;\underline{M}) = \pi_* \underbrace{F(X, H\underline{M})}$ and $H_*(X;\underline{M}) = \pi_*(X \wedge H\underline{M})$

V^{th} space $\qquad\qquad\qquad$ V^{th} space

$F(X, H\underline{M}(V)) \qquad\qquad\qquad X \wedge H\underline{M}(V)$

These are genuine spectra so $RO(G)$ graded. So really should define

$$H_\star(X;\underline{M}) \cong H_{\star+V}(S^V \wedge X; \underline{M})$$

(Careful since done virtually rather than by pairs.)

<u>Cellular Homology</u> Seeing $RO(G)$ grading directly here is hard. Interesting discussion of this ensued but was hard to transcribe.

$$\begin{array}{ccccccc} X^{n-1} \hookrightarrow X^n \longrightarrow X^n/X^{n-1} \simeq T_{n+} \wedge S^n & \longrightarrow & \Sigma X^{n-1} \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \searrow^{\delta_n} & & \downarrow \\ & & T_{n-1+} \wedge S^n = \Sigma X^{n-1}/X^{n-2} \end{array}$$

$$\delta_n \in \{T_{n+} \wedge S^n, T_{n-1+} \wedge S^n\}^G \cong \{T_{n+}, T_{n-1+}\} \cong A(T_n, T_{n-1})$$

Apply Bredon homology to the diagram, get a s.s., but it collapses to a chain complex:

$$\cdots \longrightarrow M(T_n) \xrightarrow{M(\delta_n)} M(T_{n-1}) \longrightarrow \cdots$$

N.B. degree of a map is an element of a Burnside ring $A(T_n, T_{n-1})$
Geometric interpretation of degree is tough: unlike the non-equivariant case where we count the points in the inverse image with sign. Someone should chase this down.