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Lecture 2 (2× 45 min, 7 pages)

Mackey functors & some homotopy groups.

Def: A Mackey functor is a pair of functors $M^*, M_* : \mathcal{S}et^G \to \mathcal{A}b$ s.t.

① $M^*(T) = M_*(T) =: \underline{M}(T)$

② $\underline{M}(S \amalg T) = \underline{M}(S) \oplus \underline{M}(T)$

③ $$\begin{tikzcd} S \arrow[r] \arrow[d] & T \arrow[d] \\ S' \arrow[r] & T' \end{tikzcd} \quad \text{pullback} \implies \quad \begin{tikzcd} M(S) \arrow[r] & M(T) \\ M(S') \arrow[u] \arrow[r] \arrow[ur, phantom, "G"] & M(T') \arrow[u] \end{tikzcd}$$

By ② it is determined on orbits: $\mathcal{O}_G$ ← orbit category, cat w/ obj G/H, mor $=$ G-maps

$$\mathrm{Map}_G(G/H, G/K) = (G/K)^H \neq \phi \iff H \leq {}^gK, \text{ some } g.$$

So all maps are projections $G/H \to G/K$ and conjugations & their composites.

Have to reinterpret ③ to understand a Mackey functor given in terms of its restriction to $\mathcal{O}_G$: get double coset formula. (Adams)

Ex: M a G-module gives $\underline{M}(G/H) := M^H$ "fixed point Mackey"

If $H \leq K$ then $\pi_H^K : G/H \to G/K$ induces

$$(\pi_H^K)^* : M^K \to M^H \quad \text{is inclusion} \qquad \text{(restriction)}$$

and

$$(\pi_H^K)_* : M^H \to M^K \quad \text{is} \quad \sum_{k \in K/H} k \cdot (-) \qquad \text{(transfer)}$$

E.G. $M = \mathbb{Z}$ with trivial action gives $\underline{\mathbb{Z}}$ w/ $(\pi_H^K)^* = 1$, conj $= 1$,

$tr_H^K = [K:H] \cdot (\,)$ (mult by index).

Ex: Burnside ring Mackey functor $\underline{A}(G/H) = A(H)$ (Gr. gp of finite G-sets)

restriction is just restriction,

transfer: $T \mapsto K \times_H T$.

[coordinate free version: finite G-sets over a given G-set]

Map of Mackey functors is a collection of homomorphisms

$$\underline{M}(G/H) \xrightarrow{f_H} \underline{N}(G/H)$$

commuting with structure maps.

So Maps out of $\underline{A}$: $A(H) = \langle H/H,\ H/J = tr_J^H(J/J) \mid J \subsetneq H \rangle$

so $\underline{A} \xrightarrow{f} \underline{M} \quad \mapsto f_G(G/G)$ gives $\mathrm{Mackey}(\underline{A}, \underline{M}) \cong \underline{M}(G/G)$

since $f_H(H/J) = tr_J^H f_J(J/J) = tr_J^H res_J^G f_G(G/G)$.

So $\underline{A}$ is the free Mackey functor given by something in G/G.

<u>Enrichment</u> & more freedom (same issue)

Induction & Coinduction in Mackey functors

$\mathcal{M}ackey^G \qquad \mathcal{M}ackey^H$

$$\mathcal{S}et^G \underset{\underset{G\times_H(\ )}{ind}}{\overset{i_H^* = restr}{\rightleftarrows}} \mathcal{S}et^H$$

$$(i_H^* \underline{M})(T) = \underline{M}(G \times_H T)$$

$i_{H!}$ = left adjoint to $i_H^* : \mathcal{M}ackey^G \to \mathcal{M}ackey^H$

So

$$i_{H!} \underline{M}(T) = \underline{M}(i_H^* T) \qquad \text{(Note variance swap upon precomposing)}$$

$(\mathrm{Ind}_H^G \underline{M})(T) :=$ to avoid $i_{H!}$

Ind $\dashv$ Forget $\dashv$ Co Ind — but fields disagree on terminology

$$\mathrm{Ind}_H^G i_H^* \underline{A} = \mathrm{Ind}_H^G \underline{A} : T \longmapsto \underline{A}(G \times_H i_H^* T) = \underline{A}(G/H \times T)$$ ← Call this $\underline{A}_{G/H}$

In general

$$\mathrm{Ind}_H^G i_H^* = \text{precompose w/ } G/H \times (\;)$$

$$\begin{tikzcd} \mathrm{Ind}_H^G i_H^* \underline{M}(T) = \underline{M}(G/H \times T) \arrow[d, "M_*\left(G/H\times T \xrightarrow{\text{proj}} T\right)"] \\ \underline{M}(T) \end{tikzcd}$$

so

$$\mathrm{Mackey}(\underline{A}_{G/H}, \underline{M}) \cong \underline{M}(G/H)$$ so $\underline{A}_{G/H}$ are a set of generators.

__Mackey functor valued__ hom's follow:

$$\underline{\mathrm{Mackey}}(\underline{A}_{(-)}, \underline{M}) \cong \underline{M}$$

Brief aside on homological algebra:

An Abelian cat of Mackey functors. $\underline{A}_T$ is proj, where $\underline{A}_T(S) = \underline{A}(T \times S)$. and we have enough epis from sums of these in the obvious way.

A small projective resolution: $G = C_2$, $\underline{M} = \underline{\mathbb{Z}}$.

__Notation__

$$\begin{tikzcd} \underline{M}(C_2/C_2) \arrow[d, bend right, "\text{res}"'] \\ \underline{M}(C_2/e) \arrow[u, bend right, "\text{ind}"'] \arrow[loop right, "\text{conj}"] \end{tikzcd} \qquad \underline{\mathbb{Z}}: \begin{tikzcd} \mathbb{Z} \arrow[d, bend right, "1"'] \\ \mathbb{Z} \arrow[u, bend right, "2"'] \arrow[loop right, "1"] \end{tikzcd} \qquad \underline{A}: \begin{tikzcd} \mathbb{Z}\oplus\mathbb{Z} \; \langle C_2/C_2\rangle \oplus \langle C_2/e\rangle \arrow[d, bend right, "(1\;2)"'] \\ \mathbb{Z} \arrow[u, bend right, "\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]"'] \arrow[loop right, "1"] \end{tikzcd}$$

$$\underline{A}_{C_2}: \begin{tikzcd} \mathbb{Z} \arrow[d, bend right, "\Delta"'] \\ \mathbb{Z}\oplus\mathbb{Z} \arrow[u, bend right, "\nabla"'] \arrow[loop right, "\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)"] \end{tikzcd}$$

Now we can start resolving. $\underline{\mathbb{Z}}$ is gen by 1 at C_2/C_2. So $\underline{\mathbb{Z}} \leftarrow \underline{A}$ epi hitting this.

$$\begin{tikzcd} \underline{\mathbb{Z}} & \underline{A} \arrow[l] \arrow[dr] & & \underline{A} \arrow[ll] \arrow[dl] \arrow[dr] & & \underline{A}_{C_2} \arrow[ll] & & \underline{A}_{C_2} \arrow[ll] \arrow[dr] \\ & & \underline{I} \arrow[ul] & & \underline{\mathbb{Z}}^* \arrow[ul] \arrow[ur, leftarrow] & & & & \underline{\mathbb{Z}} \arrow[ul] \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{Z} \arrow[d, bend right] \\ 0 \arrow[u, bend right] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z} \arrow[d, bend right, "2"'] \\ \mathbb{Z} \arrow[u, bend right, "1"'] \end{tikzcd}$$

so we get a periodic resolution of period 4.

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Q: $\mathrm{Ext}(\underline{\mathbb{Z}},\underline{\mathbb{Z}})$ as an algebra?

$\exists$ UCT ss starting from Mackey functor ext groups to

Q: Morita equivalence to $\mathrm{Mod}(\mathrm{End}(\underline{A}\oplus\underline{A}_{C_2}))$. What is this?

$\underline{\pi_0} S^V$: if $V^G \neq \{0\}$ the f.p. are connected so $\underline{\pi_0} S^V = \{0\}$.
But if $V^G = \{0\}$ we have $S^0 \xrightarrow{a_V} S^V$ essential. Whenever $V^H \neq \{0\}$, $i_H^* a_V = 0$. These build Krull dim in homotopy gps of spheres.

Note also $S^0 \xrightarrow{a_V} S^V$ is incl of 0-skeleton.

Let $\mathcal{F}_V = \{H \mid V^H \neq \{0\}\}$, so if $H \in \mathcal{F}_V$, $K \leq H$ then $K \in \mathcal{F}_V$
and $H \in \mathcal{F}_V \Rightarrow gHg^{-1} \in \mathcal{F}_V$,
i.e. $\mathcal{F}_V$ is a <u>family</u>.

Let $E\mathcal{F}_V \otimes \underline{M}$ = sub Mackey functor gen by $\underline{M}(G/H)$, $H \in \mathcal{F}_V$.
Then
$$\underline{\pi_0} S^V = \underline{A} / E\mathcal{F}_V \otimes \underline{A} \quad \text{gen by } a_V.$$

Smallest quotient of $\underline{A}$ which is 0 at $H \in \mathcal{F}_V$.
A couple of quick examples.

$$(E\mathcal{F} \otimes \underline{A})(G/G) = \text{sub abel gp gen by } G/H,\ H \in \mathcal{F}.$$

sign

$$\underline{\pi_0} S^\sigma = \underline{A} / E\mathcal{F}_\sigma \otimes \underline{A} : \quad \begin{tikzcd} \mathbb{Z} \arrow[d, bend right] \\ 0 \arrow[u, bend right] \end{tikzcd}$$

reduced regular

$$(\underline{\pi_0} S^{\bar\rho_G})(G/H) = \begin{cases} 0 & H \subsetneq G \\ \mathbb{Z} & H = G \end{cases}$$

have killed all proper subgps, i.e. all transfer information.

$\underline{\pi}_0 S^V$ has therefore a universal property. <u>Isotropy Separation</u>

If $\underline{\pi}_0(X) = \underline{M}$ then $\underline{\pi}_0(X \wedge S^V) = \underline{M} / E\mathcal{F}_V \otimes \underline{M}$ "localization (nullification) of $\underline{M}$ wrt the cat of Mackey functors induced up from elements of $\mathcal{F}_V$."

$$\begin{array}{ccl} \mathrm{Ind}_H^G \underline{N} \longrightarrow \Phi^{\mathcal{F}} \underline{M} & & \text{is always } 0, \quad H \in \mathcal{F} \\ \uparrow & \Big\} & \text{univ w/ this property} \\ \underline{M} & & \end{array}$$

Can copy this in Spectra via geometric fixed points. Universal space

$$E\mathcal{F}^H = \begin{cases} \phi & H \notin \mathcal{F} \\ * & H \in \mathcal{F} \end{cases}$$

$\mathcal{F} = \{e\}$ gives $E\mathcal{F} = EG$.
$\mathcal{F}_V$ gives $E\mathcal{F}_V = S(\infty V) = \varinjlim S(nV)$ (unit sphere, not 1-pt cptification)
$\mathcal{P}$ = proper subgroups gives $E\mathcal{P} = S(\infty \bar{\rho}_G)$

$$\underline{\pi}_0 S(V) \longrightarrow \underline{\pi}_0 S^0 \xrightarrow{\cdot a_V} \underline{\pi}_0 S^V \qquad \text{by cof. seq } S(V)_+ \to S^0 \to S^V$$

$$\text{“} E\mathcal{F}_V \otimes \underline{A} \text{”} \longrightarrow \underline{A} \longrightarrow \underline{A} / E\mathcal{F}_V \otimes \underline{A} \qquad (\text{Complicated after } \underline{\pi}_0)$$

(image anyway)

But taking ∞V instead, we get $S(\infty V)_+ \longrightarrow S^0 \longrightarrow S^{\infty V}$

$\wr$ complete wrt a_V $= \varprojlim / a_V^n$

$S^{\infty V} = \mathrm{colim}(S^0 \xrightarrow{a_V} S^V \xrightarrow{a_V} S^{2V} \xrightarrow{a_V} \cdots)$

so $= S^0[1/a_V]$

<u>Def</u> (Greenlees)(Carlsson)

$$E\mathcal{F}_{V+} \longrightarrow S^0 \longrightarrow \widetilde{E\mathcal{F}}_V$$

More generally $E\mathcal{F}_+ \to S^0 \to \widetilde{E\mathcal{F}}$ and this gives

$$E\mathcal{F}_+ \wedge X \longrightarrow X \longrightarrow \widetilde{E\mathcal{F}} \wedge X =: \Phi^{\mathcal{F}} X$$

Similarly a localization, nullifying everything induced up from elements of $\mathcal{F}$

$\phi^{\mathcal{F}}$ not exact, of course.

$$E\mathcal{F}_+ \wedge X \longrightarrow X \longrightarrow \tilde{E}\mathcal{F} \wedge X$$

sees family — sees everything — sees complement of family

Geometric fixed points (new notation, not $\Phi^G X$)

$$X^{gG} = (\phi^{P} X)^G \qquad \text{(genuine f.p. on a complete universe)}$$

$(\ )^G$ is an equiv of cat's on local objects, of which $\phi^{P} X$ is one.

$$\text{local obj in genuine } G\text{-spectra} \xrightarrow{(\ )^G} \text{Spectra}$$

(In spaces, f.p. = geom f.p.)

Commutation: ① $\Phi^G(\Sigma^\infty X) \simeq \Sigma^\infty X^G$ (from $\Sigma^\infty(X^G \hookrightarrow X)$)

② $\Phi^G(X \wedge Y) \simeq \Phi^G X \wedge \Phi^G Y$ (since $\tilde{E}\mathcal{F} \wedge \tilde{E}\mathcal{F} \simeq \tilde{E}\mathcal{F}$)

[Of course we need to discuss fixed points for spectra to define all this.]

G-Spectra (à la Lewis-May)

G-spaces indexed by f.d. subrep'ns of $\mathcal{U}$ &
for $V \subseteq W$

$$S^{W-V} \wedge X(V) \longrightarrow X(W)$$

Ω-spectrum: adjoints $X(V) \xrightarrow{\simeq} \Omega^{W-V} X(W)$

Given a G-Spectrum X, get an Ω G-Spectrum $(QX)(V) = \varinjlim_{W} \Omega^W X_{V \oplus W}$
The fixed points of X are fixed points of QX

$$X^G(\mathbb{R}^n) = (QX(\mathbb{R}^n))^G$$

For $X = \Sigma^\infty S^0$, when $X(V) = S^V$, we get $QS^0 = \varinjlim \Omega^V S^V$

which has more fixed points than you'd think:

$$\pi_0 (QS^0)^G = \varinjlim \pi_0 (\Omega^V S^V)^G = \varinjlim [S^V, S^V]^G = A(G)$$

This is why fixed points is so complicated. The failure of $A(G)$ to be $\mathbb{Z}$ is the information coming from transfers.

Restrict now to $G \twoheadrightarrow C_2$. So think just about C_2.

$$\begin{tikzcd}
C_{2+} \arrow[r, "\pi"] \arrow[d, equal] & S^0 \arrow[r, "a_\sigma"] \arrow[d, equal] & S^\sigma \arrow[d, equal] \\
(G/\ker)_+ \arrow[r] & (G/G)_+ \arrow[r] & S^\sigma
\end{tikzcd}$$

sign rep'n for G

$$\begin{tikzcd}
{[C_{2+} \xrightarrow{\pi} S^0, X]^{C_2}} = & {[S^0, X]^{C_2}} \arrow[d] = & \underline{\pi}_0(X)(C_2/C_2) \arrow[d, "res^{C_2}_e"] \\
& {[C_{2+}, X]^{C_2}} = & \underline{\pi}_0(X)(C_2/e)
\end{tikzcd}$$

Continuing (✯)

$$\begin{tikzcd}
\pi_\sigma(X)(C_2/C_2) \arrow[r, "\cdot a_\sigma"] & \pi_0(X)(C_2/C_2) \arrow[r, "res^{C_2}_e"] & \underline{\pi}_0(X)(C_2/e) \arrow[r, bend left, "so"] & \pi_{\sigma-1}(X)(C_2/C_2)
\end{tikzcd}$$

$$\boxed{\ker(res^{C_2}_e) = Im(- \cdot a_\sigma)}$$

Alternatively, smashing with the cofiber sequence, we get

$$\underline{\pi}_0(X)(C_2/e) \xrightarrow{tr^{C_2}_e} \underline{\pi}_0(X)(C_2/C_2) \xrightarrow{\cdot a_\sigma} \underline{\pi}_{-\sigma}(X)(C_2/C_2) \xrightarrow{res} \underline{\pi}$$

so $\boxed{\ker(- \cdot a_\sigma) = Im(tr^{C_2}_e)}$

Finally, curved map in (✯) gives a dim shifting tr.
Transfer $(S^0 \to C_{2+}) \wedge S^\sigma$ is $S^\sigma \to C_{2+} \wedge S^\sigma \cong C_{2+} \wedge S^1$
and $\wedge S^{-1}$ gives $S^{\sigma-1} \to C_{2+}$. "signed transfer" = transfer from an $RO(G)$ graded stem, $1-\gamma$, instead of $1+\gamma$ like usual transfer.

The two boxed statements are the key to tomorrow's computations.