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Lecture 1 (2×45 min) 7 pages

Computations in Equivariant Stable Homotopy

One group at a time.

Basics today. Computations, mainly tomorrow.

Rmk: G finite. Extension to cpt Lie?

Def: G set X is a set X w/ homomorphism $G \to \mathrm{Aut}(X)$

$\mathcal{S}et^G$: obj = G-sets
mor = G-maps ($X \xrightarrow{f} Y$ s.t. $f(gx) = gf(x)$ $\forall g \in G, x \in X$)

$\left[\mathcal{S}et^G(X,Y) = \mathcal{S}et(X,Y)^G \text{ w/ conj action on } \mathcal{S}et(X,Y)\right]$

Given $x \in X$, the orbit $G \cdot x = \{g \cdot x \mid g \in G\}$,
Stabilizer $\mathrm{Stab}(x) = \{h \in G \mid h \cdot x = x\}$

Prop: $G/\mathrm{stab}(x) \xrightarrow{\pi} G \cdot x$ is a bijection of G-sets
$[g] \longmapsto g \cdot x$

Index everything by G-sets where we'd index by $\mathbb{N}$ non-equivariantly.

Space = simplicial set here.

Def: A simplicial G-set is a simplicial object in $\mathcal{S}et^G$ (finite G-sets) or $\mathcal{E}ns^G = \mathcal{M}aen^G$ (all G-sets)
$s\mathcal{M}aen^G$.

Or, G-object in $s\mathcal{S}et$, and they're the same.

Ex: If X is a G-space, i.e. space with cts action of G, then
$\mathrm{Sing}(X)_n = \mathrm{Map}(\Delta^n, X)$ defines a simplicial G-set.

Geometric realization of a simplicial G-set is a G-space. This provides a functor

$$s\mathcal{M}aen^G \longrightarrow \mathcal{T}op^G$$

If we replace (in Δ^n) the non-degen n-simplex by an orbit, ...

Def: A G-CW complex is a space which can be built by iteratively attaching cells of the form $T \times D^n$, T a G-set, along their boundary. (Aesthetically simpler than writing $G/H \times D^n$ - functoriality in G is more evident.)

(Note attaching maps are G-maps.)

Classically Sing & Geom realization are Quillen inverses. This holds here.

If $H \leq G$ the restriction functor $i_H^*: \mathcal{S}et^G \longrightarrow \mathcal{S}et^H$

$\mathcal{T}op^G \longrightarrow \mathcal{T}op^H$

etc.

Left adjoint of i_H^* is induction: $X \longmapsto G \underset{H}{\times} X = G \times X / (gh, x) \sim (g, hx)$ $\forall g, h, x$

So

$$Map_G(G \underset{H}{\times} X, Y) \cong Map_H(X, i_H^* Y)$$

$$\left(G \underset{H}{\times} X \xrightarrow{f} Y \right) \longmapsto \left(X \cong e \times X \hookrightarrow G \times X \to G \underset{H}{\times} X \xrightarrow{f} Y \right)$$

Aside:

$\exists$ also a right adjoint, coinduction $F_H(G, X)$.

Note As sets $G \underset{H}{\times} X$ is a disjoint union of G/H copies of X

$F_H(G, X)$ — " — product — " ———— .

Build a G-object from an H-object by taking some symm mon. product. (In Spectra, can use smash product & get norms)

For a G-space X, $\pi_0^G(X) = \pi_0(X^G)$.

if $H \leq G$ then $X^G \hookrightarrow X^H$, inducing map on π_0. Better,

$$\underline{\pi_0}(X) : (\mathcal{S}et^G)^{op} \longrightarrow \cancel{\mathcal{S}et}\ \mathcal{M}an \quad \text{by} \quad \underline{\pi_0}(X)(T) = \pi_0(Map_G(T, X))$$

usually called a <u>coefficient system</u>.

$$\underline{\pi}_0(X)(G/H) = \pi_0 \, Map_G(G \times_H *, X) = \pi_0 \, Map_H(*, X) = \pi_0(X^H)$$

<u>Def</u>: A G-map $X \xrightarrow{f} Y$ is a weak equivalence if f^H is a wk-equiv $\forall H \leq G$.

<u>Def</u>: $\underline{\pi}_n(X)(T) = [T \times S^n, X]_G$ pointed, i.e. $[T_+ \wedge S^n, X]_G$
where basept $x \in X$ is G-fixed.

$[\,,\,]_G$ = homotopy classes of G-maps = π_0 of a function space (pointed!)

[Worry: what if X has more G components than H components.]
[Some controversy erupted at this point.]
[Stabilization erases the controversy.]

— BREAK —

For stabilization, we need the <u>Spanier-Whitehead Category</u>
Spectra, of all sorts, are fattenings of this.

<u>Obj</u>: finite G-CW complexes
<u>Mor</u>: "stable homotopy classes of maps"

} has dualizability up to shift, so add negative spheres, & limits & colimits.

Let $\mathcal{U}$ be a universe for G, then

(1) $$\{X, Y\}^G_{\mathcal{U}} = \varinjlim_{\substack{V \subset \mathcal{U} \\ \text{f.d.}}} [S^V \wedge X, S^V \wedge Y]_G \quad , \quad S^V = V \cup \{\infty\}, \text{ 1 point compactification.}$$

cofinally, $V \subset \mathcal{U}$ indep of how V sits in $\mathcal{U}$

<u>Note</u>: trivial 1-dim rep'n $1 \leq \mathcal{U}$, so $\mathbb{R}^\infty \subset \mathcal{U}$, so we get the usual properties of stabilization:
- these are ab gps
- composition is linear
- $\{X, Y \vee Z\}^G_{\mathcal{U}} \cong \{X, Y\}^G_{\mathcal{U}} \times \{X, Z\}^G_{\mathcal{U}}$

so $Y \vee Z$ is both product & coproduct now.

Can also introduce integer shifts

$$\{X,Y\}^G_{\mathcal{U},n} = \begin{cases} \{S^n \wedge X, Y\}^G_{\mathcal{U}} & n \geq 0 \\ \{X, S^n \wedge Y\}^G_{\mathcal{U}} & n \leq 0 \end{cases}$$

Get a lot of other things, depending on choice of $\mathcal{U}$.

(2) if G/H embeds in $\mathcal{U}$, $\exists$ stable map $\{S^0, G/H_+\}^G_{\mathcal{U}}$ ← sic
which can be used to show

$$\{X, G_+ \wedge_H Y\}^G_{\mathcal{U}} \cong \{i_H^* X, Y\}^G_{\mathcal{U}}$$

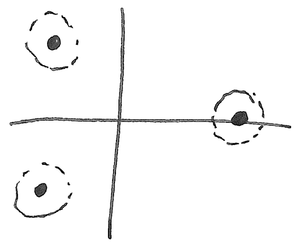
To understand this map, think about algebra: for $m \in M$ we can get a fixed point $\sum_{g \in G} gm$.

Would like to send basept to sum of points in its orbit & stabilization is needed so we can add.

Picture for $H = \{e\} \leq G = C_3$. Say

$$\begin{array}{ccc} & & V \\ & \nearrow & \downarrow \\ C_3 & \hookrightarrow & \mathcal{U} \end{array}$$

Reps/$\mathbb{R}$: triv & 2-dim are the irreps.



$C_3 \hookrightarrow \mathbb{C} = V$

Choose equivariant nbood & use Thom-Pontrjagin map

$$S^V \longrightarrow C_{3+} \wedge S^V$$

i.e. a stable map $S^0 \xrightarrow{tr} C_{3+}$

Q: why S^V not S^3?

Ans: Can always write target as $G_+ \wedge_H S^{i_H^* V}$ and nbood of f.p. is H-equivariantly homeo to nbood of 0.

(Key: freeing up by G_+, it ~~mak~~ gives the same result.) ($G_+ \wedge_H S^{i_H^* V} \cong G/H_+ \wedge S^V$)

(Verbal description of the isomorphism, no time to write)

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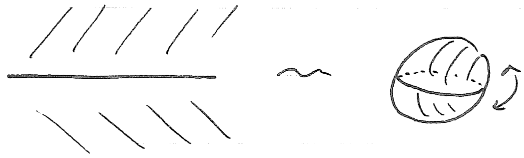
<u>Cor</u>: (Wirthmüller Isomorphism) Induction & Coinduction are stably equivalent.
(Just as coproduct & product.)

Stabilization: chose a universe. Contained trivial. Got Wirtmüller, etc. Forced equivalences. To choose universe, decide which isomorphisms one wants to enforce. Getting certain kinds of colimits as limits.

<u>Ex</u>: $\{S^0, G/H_+\}^G_{\mathcal{U}}$ is now more than a point.

<u>Ex</u>: $[S^\rho, S^\rho]^{C_2}$, ρ = regular rep'n, i.e. $\mathbb{C}$ w/ $\overline{(\ )}$.
($1 \subset \rho$ gives group structure)

$C_{2+} \wedge S^1 \longrightarrow S^1 \longrightarrow S^\rho$ is a C_2-equivariant cof. sequence
$\searrow C_{2+} \wedge S^2$ } continue Barratt-Puppe sequence



$[C_{2+} \wedge S^1, S^\rho]^{C_2} \longleftarrow [S^1, S^\rho]^{C_2} \longleftarrow [S^\rho, S^\rho]^{C_2} \longleftarrow [C_{2+} \wedge S^2, S^\rho]^{C_2} \longleftarrow [S^2, S^\rho]^{C_2}$

$[C_{2+} \wedge S^1, S^\rho]^{C_2} \cong [S^1, i_e^* S^\rho] = [S^1, S^2] = 0$

$[S^1, S^\rho]^{C_2}$: fixed S^1 must lie in fixed $(S^\rho)^{C_2} = S^1$, so $= [S^1, S^1] = \mathbb{Z}$

$[C_{2+} \wedge S^2, S^\rho]^{C_2} \cong [S^2, i_e^* S^\rho] = [S^2, S^2] \cong \mathbb{Z}$

$[S^2, S^\rho]^{C_2} = [S^2, S^1] = 0$ as in

So $[S^\rho, S^\rho]^{C_2} \cong \mathbb{Z} \oplus \mathbb{Z}$

Key points (1) $S^\rho \longrightarrow C_{2+} \wedge S^2$ is $S^\rho \xrightarrow[tr]{} C_{2+} \wedge S^\rho$

So tr takes underlying information & produces fixed point information.

(2) The $S^1 \to S^\rho$ is $(S^\rho)^{C_2} \to S^\rho$ so it induced restriction to fixed points.

$\rightarrow$

finite G cpt Lie G

<u>Thm</u> (Segal, tom Dieck) $\pi_0^S(S^0) = A(G)$

$= $ Groth. gp of $\mathcal{S}et^G$

$\{S^0, S^0\}^G_{\infty\rho_G}$

Each finite G set is recording an induced piece:

$$A(G) = \mathbb{Z}\langle [G/H] \mid H \leq G \text{ up to conj} \rangle$$

$$\uparrow$$

$$tr_H^G(H/H)$$

$$\parallel$$

$$G \underset{H}{\times} (H/H)$$

$\exists$ more structure: finite G-sets are self dual,

$$\{G/H_+ \wedge X, Y\}^G_{\mathcal{U}} \cong \{X, G/H_+ \wedge Y\}^G_{\mathcal{U}}$$

because both are just $\{X, Y\}^H_{\mathcal{U}}$ so long as G/H embeds in $\mathcal{U}$.

$$\Rightarrow \{T_+ \wedge X, Y\}^G_{\mathcal{U}} \cong \{X, T_+ \wedge Y\}^G_{\mathcal{U}}$$

So $\underline{\{X,Y\}}(T) := \{T_+ \wedge X, Y\}^G_{\mathcal{U}}$ a functor on finite G-sets in <u>two</u> different ways, one covariant, one contravariant.

I.E. <u>A Mackey functor</u>: two functors M_*, M^* s.t.

1. $M^*(T) \cong M_*(T) = M(T)$
2. $M(T \amalg S) \cong M(T) \oplus M(S)$
3. If
$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & T \arrow[d, "\beta"] \\ S' \arrow[r, "\delta"'] & T' \end{tikzcd}$$
is a pullback then
$$\begin{tikzcd} M(S) \arrow[r, "M_*(\alpha)"] & M(T) \\ M(S') \arrow[u, "M^*(\gamma)"] \arrow[r, "M_*(\delta)"'] & M(T') \arrow[u, "M^*(\beta)"'] \end{tikzcd}$$

③ expresses the two ways to decompose
$$(G/H) \times (G/K)$$

N.B. Only needed the Wirthmüller isomorphism, that induction & coinduction have been made equal.