Homotopical approaches to group actions

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Problem session:

I finite group actions:

(1) Does any rk 2 grp G act freely on $X \simeq S^n \times S^m$ for $n, m > 0$?
 f.t. cpx.

- True if $Qd(p)$ not subquotient of G.
- True if $|G|$ odd.

(2) Determine if $Qd(p) = (\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes SL_2(\mathbb{F}_p)$ acts freely on some $S^n \times S^m$. p odd.

(if $Qd(p)$ acts on S^n, then rk 2 isotropy.

(3) X $H^*(X;\mathbb{Z})$ has "2 poly parameters"

?∃
$$\begin{tikzcd} S^n \times S^m \arrow[r] & E \arrow[d] \\ & X \end{tikzcd}$$
where $E \simeq$ f.d. cpx.

(4) Given $X \simeq S^n$ G acts w/ periodic isotropy, when can we find a smooth action?
What about Σ_5?

Is the following equiv

$\left\{ \begin{array}{l} G \text{ action w/} \\ \text{effective Euler class} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} G \text{ action on } X \simeq S^n \text{ finite} \\ \text{w/ periodic isotropy} \end{array} \right\}$

Known for Σ_5 Pamuk-Hambleton-Yalçin

(5) Set up the notion of an equiv poincaré duality cpx and do surgery using the orbit cat.

(6) Does any G act freely + homologically triv. on some $X \simeq S^{n_1} \times \dots \times S^{n_k}$ (smoothly).

(7) When does a rk 2 grp Γ of ft. vcd act freely on some

$$X \simeq S^n \times S^m \times \mathbb{R}^k \ ?$$

Co-compactly?

(8) $rk(G) = \max\{n \mid (\mathbb{Z}/p)^n \leq G \quad p \text{ prime}\}$

$h(G) = \min\{k \mid G \text{ acts freely on some } X \simeq S^{n_1} \times \dots \times S^{n_k}\}$.

$rk(G) = h(G)$.

(9) [G. Carlsson] $G = (\mathbb{Z}/p)^r$ acts freely on a conn ft cpx X then

$$\sum_{i=0}^{\dim X} \dim H^i(X; \mathbb{F}_p) \geq 2^r$$

[Note Thm (Hanke) $(\mathbb{Z}/p)^r$ acting freely on $X \simeq S^{n_1} \times \dots \times S^{n_k}$ $p > 3\dim X$ then $r \leq k$.]

(10) If $(\mathbb{Z}/2)^r$ acts freely on a f.t. X with mod 2 cohom gen by 1-dim classes then

$$r \leq 2\dim H^1(X;\mathbb{F}_2)$$

(11) If X f.t. 1-conn.

Is $H^*(B\mathrm{Aut}_1(X);\mathbb{Q})$ of

f.t. transcendence degree?

[Note: OK if X rationally elliptic].

(12) G f.t. grp acting effectively on a

2-dim contractible CW-cpx

Does it have a fixed pt?

(13) G f.t. Cpx reflection grp acting on $\mathbb{C}^n$.

then $M(A) = \mathbb{C}^n \setminus \cup H$ is a $K(\pi,1)$

H hyperplanes.

II Group Cohomology

(14) k char p, S simple module in the princ. block of kG.
then is $H^n(G, S) \neq 0$ for some $n \geq 0$?

[still open? nucleus?].

(15) $H^1(G; \mathbb{F}_p) \neq 0 \Rightarrow H^{2n}(G; \mathbb{F}_p) \neq 0 \quad \forall n \geq 0$

(16) Does there exist $N > 0$ s.t. if
$H_i(\cancel{X} G; \mathbb{Z}) = 0 \quad 1 \leq i \leq N$ then $G = \{1\}$?

[Note $H_i(M_{23}; \mathbb{Z}) = 0 \quad 1 \leq i \leq 4$].

(17) Under what conditions does a map of f.t. gps induce an inseparable isogeny induce an inseparable isogeny between the mod p cohom varieties?
If p is odd, is it true that this cond., together w. cond that the Sylow p-subgps have the same order, inducing an iso in cohomology?

i.e. p odd is $G_1 \to G_2$ w same p-Sylow ??
~~same~~ iso on p-fusion $\Longleftrightarrow$ iso on mod p cohom varieties

(18) Compute $H^*(GL_n(\mathbb{F}_p);\mathbb{F}_p)$???

In a range?

(19) Calculate $H^*(PU(n);\mathbb{F}_p)$ $p \mid n$.

(20) Given

$$1 \to (\mathbb{Z}/2)^r \to U \to (\mathbb{Z}/2)^s \to 1$$

central does the EMSS collapse at E_3?

$$\mathrm{Tor}_{H^*(K((\mathbb{Z}/2)^r,2))}(\mathbb{F}_2, H^*((\mathbb{Z}/2)^s)) \Rightarrow H^*(U;\mathbb{F}_2)$$

(21) $H^*(BSp_\infty(\overset{\mathbb{Z}_p}{\cancel{\mathbb{Z}/p}});\mathbb{F}_p) = ?$

[See recent paper of A. Putman].

(22) If $|A_p(G)|$ contractible then it is equivariantly contractible

(23) Does $H^*(G;k)$ always have an associated prime $\mathfrak{p}$ s.t.
Krull dim of $H^*(G)/\mathfrak{p}$ = depth.

(24) If $H^*(G)$ is detected on elt ab gps can you figure out its depth.

$$\operatorname{depth} \lim_{E \in \mathcal{A}_p(G)}{}^{0} H^*(E) = ??$$

(25) Under what conditions on $N \rtimes H$ does $E_2 = E_\infty$ in LHS SS.

N finite p-gp with central series for which each layer is a proj $\mathbb{F}_p H$-module!

(26) Give necessary + sufficient cond for existence of a G-complex X s.t.

$$\hat{H}^*(X;\mathbb{Z}) \cong M.$$