BACKGROUND ON CATEGORY THEORY

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1. Categories and functors

We have the notion of a category, and of functors between categories.

Exercise 1.1. Look up the following notions: a category, a (covariant or contravariant) functor between categories, a morphism between functors, a full subcategory.

We will talk about abelian categories; In those, the sets of morphisms are abelian groups, and we have the notions of zero object, direct sums of objects, kernel, image, and so on. When we consider functors between abelian (or additive) categories, we (usually, unless otherwise said) only consider additive functors.

Exercise 1.2. Look up the definition of an abelian category.

Example 1.3. The category $R - mod$ of left modules over a unital ring R is an abelian category. If R is Noetherian, the full subcategory of $R - mod$ consisting of finitely generated modules is also abelian.

Example 1.4. Let G be an l-group. The category of smooth complex representations of G is an abelian category (it is an object of study in p-adic representation theory).

Example 1.5. Let A be an idempotented algebra. Consider the category $A - mod$ of non-degenerate left modules over A. It is an abelian category. An example of an idempotented algebra which is relevant to us is the algebra $\mathcal{H}(G)$ of smooth, compatly supported measures on an l-group G (it is an algebra w.r.t. convolution). Another example is the algebra $S(X)$ of smooth, compactly supported functions on an l-space X (it is an algebra w.r.t. pointwise product).

Mitchells embedding theorem asserts that given a small abelian category, it is possible to embed it into the category $R - mod$ for some unital ring R. Here, "embed" means to find a fully faithful and exact functor from it to $R - mod$. So in all results of diagrammatic nature, we can pretend that our abelian category is the category of modules over a ring.

Inside a category we have the notion of two objects being "the same"; Technically, isomorphic.

Exercise 1.6. Which "concrete" invariant classifies isomorphism classes in: The category of vector spaces? The category of finitely generated abelian groups? The category of finite-dimensional complex representations of a finite group? The category of finite-dimensional complex vector spaces together with an endomorphism?

Recall the very important Yoneda lemma:

Theorem 1.7. *Let $\mathcal{A}$ be a category (an additive category). Consider the functor $\mathcal{A} \to Fun(\mathcal{A}^{op}, Sets)$ ($\mathcal{A} \to Fun(\mathcal{A}^{op}, Ab)$) given by $M \mapsto (N \mapsto Hom(N, M))$. Then this functor is fully faithful.*

So, an object is determined by how other objects map into it.

2. Equivalence of categories

Important question: When are two categories $\mathcal{A}$ and $\mathcal{B}$ "the same"?

Usual answer: When there is an equivalence between them, which is a functor $F : \mathcal{A} \to \mathcal{B}$ which is fully faithful and essentially surjective. Being fully faithful means that $Hom(X, Y) \to Hom(F(X), F(Y))$ is a bijection for all $X, Y \in \mathcal{A}$. Being essentially surjective means that any object of $\mathcal{B}$ is isomorphic to F applied to some object of $\mathcal{A}$.

Since categories "see" only relations between objects, it is possible that two categories will "look" (superficially) very different, but will be in fact equivalent.

Example 2.1. The category of smooth representations of an l-group G is equivalent to the category of non-degenerate modules over the algebra $\mathcal{H}(G)$. The category of sheaves of vector spaces on an l-space X is equivalent to the category of non-degenerate modules over the algebra $S(X)$.

Example 2.2. The category of locally constant sheaves on a nice connected pointed topological space is equivalent to the category of representations of its fundamental group.

Exercise 2.3. Recall some equivalences of categories.

3. Exact functors, Projective objects

Exercise 3.1. Recall the notion of an exact sequence, a short exact sequence (in an abelian category).

Example 3.2. In the category $\mathbb{Z} - mod$, we have the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ where the second arrow is multiplication by 2, and the third arrow is the projection.

Exercise 3.3. Recall the notion of an exact functor between abelian categories, and of a left exact and right exact functor.

Example 3.4. In any abelian category $\mathcal{A}$, for an object X, the functor $\mathcal{A} \to Ab$ given by $Y \mapsto Hom(X, Y)$ is left exact. If this functor is actually exact, the object X is called projective.

Exercise 3.5. Show that in $R - mod$ (R a unital ring), an object is projective if and only if it is a direct summand of a free object.

Example 3.6. Let R be commutative unital. In $\mathcal{A} = R - mod$, for an object X, the functor $\mathcal{A} \to \mathcal{A}$ given by $Y \mapsto X \otimes_R Y$ is right exact. If this functor is actually exact, the object X is called flat.

Exercise 3.7. Show that a projective module is flat. Show that $\mathbb{Q}$ is a flat $\mathbb{Z}$-module which is not projective.

4. Adjunction

Given two functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$, an adjunction between them is an isomorphism $Hom(FX, Y) \cong Hom(X, GY)$ for all $X \in \mathcal{A}, Y \in \mathcal{B}$, which is "functorial" in X and Y.

Exercise 4.1. Understand the exact definition (i.e. what "functorial" means).

The roles of F and G are not symmetric; We say that F is the left adjoint of G (and G is the right adjoint of F).

Exercise 4.2. Let F be the left adjoint of G. Show that we get morphisms of functors $Id \to G \circ F$ and $F \circ G \to Id$. Show that the composites $F \to F \circ G \circ F \to F$ and $G \to G \circ F \circ G \to G$ are the identity morphisms. Show that conversely, if we have morphisms $Id \to G \circ F$ and $F \circ G \to Id$ with this conditions, then F becomes naturally left adjoint to G.

If a functor F admits a right adjoint G, then one can see (using Yonedas lemma) that G is unique, up to a unique isomorphism. In other words, for a functor F, the question "Does F admit a right adjoint" is well-posed, with answer being "yes" or "no" (i.e. if the answer is "yes", we "know" the right adjoint, it is unique in the only possible sense).

Exercise 4.3. Formulate what does it mean for the saught for adjoint functor to be unique up to a unique isomorphism.

Exercise 4.4. Use Yonedas lemma to proof that an adjoint functor is unique up to a unique isomorphism.

Example 4.5. Let $\mathcal{C}$ be the category of Normed real vector spaces, $\mathcal{D}$ the category of complete normed real vector spaces (a.k.a. Banach spaces). In both cases, morphisms are continuous linear maps. We have the inclusion functor $\mathcal{D} \to \mathcal{C}$. It has a left adjoint, which is the functor of completion. So the notion of a left adjoint functor allows us, once we formulate what is a complete normed real vector space, to have a well-defined notion of completion, even before we try to complete a normed real vector space!

Example 4.6. Let R be commutative unital. For $\mathcal{A} = R - mod$, The functor $Hom(X, \cdot)$ is right adjoint to the functor $X \otimes_R \cdot$ (both considered as functors $\mathcal{A} \to \mathcal{A}$).

Example 4.7. This is a non-additive example. The functor $Vect \to Set$, from the category of vector spaces to that of sets, admits a left adjoint $Set \to Vect$, which associates to a set the vector space which has this set as a basis.

Exercise 4.8. Verify these examples, and come up with some of your own.

If F has a right (left) adjoint, it is right (left) exact, and in addition it commutes with direct sums (direct products). This is very useful.

Exercise 4.9. Prove the above claims.

There are sometimes converse theorems; That in "good" cases, a functor which is right exact and commutes with direct sums admits a right adjoint (so called "adjoint functor theorems"). For example:

Theorem 4.10. *Let R be a unital ring, and $\mathcal{A}$ an abelian category. Let $F : R - mod \to \mathcal{A}$ be a right exact functor, which commutes with direct sums. Then F admits a right adjoint $G : \mathcal{A} \to R - mod$.*

Exercise 4.11. Prove this theorem.

5. Projective generators

From a unital ring R, we get an abelian category $\mathcal{A} = R - mod$. We can ask the converse: How to relate abelian categories to categories of modules? Let $\mathcal{A}$ be an abelian category which has direct sums, and $P \in \mathcal{A}$ an object. Then, we have the ring $End(P) := Hom(P, P)$ (it is a ring w.r.t. composition). In addition, for every object $M \in \mathcal{A}$, the abelian group $Hom(P, M)$ in fact carries the structure of an $End(P)^{op}$-module (by precomposing). So, we get in this way a functor $\Phi : \mathcal{A} \to End(P)^{op} - mod$. This is the main observation. Now we can ask when is this functor, say, an equivalence of categories.

Theorem 5.1. *The functor constructed above is an equivalence of categories if and only if P is a compact projective generator.*

The terminology is as follows. We already defined P being projective to mean that $Hom(P, \cdot)$ is exact. P being compact means that $Hom(P, \cdot)$ commutes with direct sums. P being a generator means that if $Hom(P, M) = 0$, then $M = 0$.

Exercise 5.2. Prove this theorem.

Exercise 5.3. Consider the abelian category $R - mod$ (R unital). Let $M \in R - mod$. Denote by $\Psi : R - mod \to Ab$ the functor $\Psi(N) = Hom(M, N)$. Show:

- Ψ commutes with filtered colimits if and only of M is finitely presented.
- If M is finitely generated, Ψ commutes with direct sums (to commute with direct sums is weaker than to commute with filtered colimits).
- If M is projective and Ψ commutes with direct sums, then Ψ commutes with arbitrary colimits (hence, in particular, a projective finitely generated module is finitely presented).

Example 5.4. Let R be unital. In the category $R - mod$, the obejct R is a compact projective generator. Of course, the equivalence that we get in this way is just $R - mod \approx R - mod$ in the boring way. To do something a bit more interesting, note that the direct sum of two compact projective generators is also a compact projective generator. Hence, in $R - mod$ we have the compact projective generator R^n. Using it, we establish an equivalence of categories $R - mod \approx M_n(R) - mod$.

Example 5.5. Let A be an idempotented algebra (so $A - mod$ is the category of non-degenerate modules). Let $e \in A$ be an idempotent. Then Ae is a compact projective object in $A - mod$. Furthermore, $End(Ae)^{op}$ is isomorphic to eAe (this is a sublagebra of A, which does admit a unit). So if Ae is a generator, which means that $eM = 0$ implies $M = 0$ for $M \in A - mod$, we will get an equivalence of categories $A - mod \approx eAe - mod$.

Exercise 5.6. Show that in the example above, Ae is a generator if and only if $A = AeA$.

Exercise 5.7. Show that in the example above, Ae is a generator if and only if $eM \neq 0$ for every irreducible A-module M. Hint: Show that any module admits an irreducible subquotient (you will have to use Zorns lemma).

Example 5.8. Let A be the algebra of $\mathbb{N} \times \mathbb{N}$-matrices over a unital ring R, whose all entries, except finitely many ones, are 0. It is an idempotented algebra. Let $e \in A$ denote the idempotent which is 1 at the 1×1 place, and 0 everywhere else. Then it is easy to verify that $A = AeA$. Furthermore, eAe is clearly isomorphic to R. Thus, we get an equivalence of categories $A - mod \approx R - mod$.